

# Viscous Aubry-Mather theory and the Vlasov equation

Ugo Bessi\*

## Abstract

The Vlasov equation models a group of particles moving under a potential  $V$ ; moreover, each particle exerts a force, of potential  $W$ , on the other ones. We shall suppose that these particles move on the  $p$ -dimensional torus  $\mathbf{T}^p$  and that the interaction potential  $W$  is smooth. We are going to perturb this equation by a Brownian motion on  $\mathbf{T}^p$ ; adapting to the viscous case methods of Gangbo, Nguyen, Tudorascu and Gomes, we study the existence of periodic solutions and the asymptotics of the Hopf-Lax semigroup.

## Introduction

The Vlasov equation models the motion of a group of particles under the action of a time-dependent potential  $V$  and a mutual interaction  $W$ . For definiteness, we shall suppose that the particles move on the torus  $\mathbf{T}^p := \frac{\mathbf{R}^p}{\mathbf{Z}^p}$ ; we put on the position and velocity space  $\mathbf{T}^p \times \mathbf{R}^p$  the coordinates  $(x, v)$  and we suppose that, at time  $t$ , the particles are distributed on  $\mathbf{T}^p \times \mathbf{R}^p$  according to a probability measure  $f_t$ . Then, the Vlasov equation has the form

$$\partial_t f_t + \langle v, \partial_x f_t \rangle = \operatorname{div}_v (f_t \partial_x P_t) \quad (VL)_\infty$$

where

$$P(t, x) = V(t, x) + \int_{\mathbf{T}^p \times \mathbf{R}^p} W(x - x') df_t(x', v')$$

and  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbf{R}^p$ . Since [7], one looks for weak solutions of  $(VL)_\infty$ ; in other words, given an initial distribution  $f_0$ , one looks for a continuous curve of probability measures  $f_t$  satisfying

$$\int_{\mathbf{T}^p \times \mathbf{R}^p} \phi(0, x, v) df_0(x, v) + \int_0^{+\infty} \int_{\mathbf{T}^p \times \mathbf{R}^p} [\partial_t \phi(t, x, v) - \langle v, \partial_x \phi(t, x, v) \rangle + \langle \partial_x P(t, x), \partial_v \phi(t, x, v) \rangle] df_t(x, v) = 0$$

for all  $\phi \in C_0^\infty([0, +\infty) \times \mathbf{T}^p \times \mathbf{R}^p)$ .

Our hypotheses on  $V$  and  $W$  are the following:

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\* Dipartimento di Matematica, Università Roma Tre, Largo S. Leonardo Murialdo, 00146 Roma, Italy.  
email: [bessi@matrm3.mat.uniroma3.it](mailto:bessi@matrm3.mat.uniroma3.it) Work partially supported by the PRIN2009 grant "Critical

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- 1)  $V \in C(\mathbf{T}, C^3(\mathbf{T}^p))$ , and
- 2)  $W \in C^3(\mathbf{T}^p)$ . Thus,  $W$  lifts to a  $C^3$  function on  $\mathbf{R}^p$ ,  $\mathbf{Z}^p$ -periodic; we shall also suppose that  $W(x) = W(-x)$  and that  $W(0) = 0$ .

A recent idea (see [11], [12]) is to view  $(VL)_\infty$  as a Lagrangian system in the space of measures; indeed, it is possible to define what it means for a curve  $\mu_t$  of probability measures on  $\mathbf{T}^p$  to minimize the Lagrangian action

$$\int_{t_0}^{t_1} \left[ \frac{1}{2} \|\dot{\mu}_t\|^2 + \int_{\mathbf{T}^p} V(t, x) d\mu_t + \frac{1}{2} \int_{\mathbf{T}^p \times \mathbf{T}^p} W(x - x') d(\mu_t \times \mu_t)(x, x') \right] dt. \quad (1)$$

The advantages are that one can use the tools of Lagrangian dynamics (Aubry-Mather theory, Hamilton-Jacobi equations, minimal characteristics, etc...) albeit in the difficult "differential manifold" of probability measures.

In this paper, we are going to adapt to the viscous case an older approach: following [7], we jury-rig a fixed point theorem to the viscous Mather theory of [13]. Let us briefly outline what we are doing in the case of periodic orbits.

We let  $\psi(t, x)$  be a continuous family of densities, periodic in time; in other words, we ask that

- d1)  $\psi \in C(\mathbf{T} \times \mathbf{T}^p)$
- d2)  $\psi \geq 0$
- d3)  $\int_{\mathbf{T}^p} \psi(t, x) dx = 1$  for all  $t \in \mathbf{T}$ .

Let us define

$$P_\psi(t, x) = V(t, x) + \int_{\mathbf{T}^p} W(x - x') \psi(t, x') dx'$$

and, for  $c \in \mathbf{R}^p$ , let us set

$$\mathcal{L}_{c, \psi}: \mathbf{T} \times \mathbf{T}^p \times \mathbf{R}^p \rightarrow \mathbf{R}, \quad \mathcal{L}_{c, \psi}(t, x, \dot{x}) = \frac{1}{2} |\dot{x}|^2 - \langle c, \dot{x} \rangle - P_\psi(t, x)$$

$$H_\psi: \mathbf{T} \times \mathbf{T}^p \times \mathbf{R}^p \rightarrow \mathbf{R}, \quad H_\psi(t, x, p) = \frac{1}{2} |p|^2 + P_\psi(t, x).$$

We have the following.

**Theorem 1.** *Let  $c \in \mathbf{R}^p$  and let  $\beta > 0$ . Then, there is a couple of functions*

$$\rho_\beta \in C^1(\mathbf{T} \times \mathbf{T}^p) \cap C(\mathbf{T}, C^2(\mathbf{T}^p)), \quad u_\beta \in C^1(\mathbf{T}, C^1(\mathbf{T}^p)) \cap C(\mathbf{T}, C^3(\mathbf{T}^p))$$

and  $\bar{H}_\beta(c) \in \mathbf{R}$  such that  $\rho_\beta$  satisfies points d1)-d3) above and

$$\frac{1}{2\beta} \Delta u_\beta + \partial_t u_\beta - H_{\rho_\beta}(t, x, c - \partial_x u_\beta) + \bar{H}_\beta(c) = 0, \quad (HJ)_{\rho_\beta, per}$$

$$\frac{1}{2\beta} \Delta \rho_\beta - \operatorname{div}[\rho_\beta \cdot (c - \partial_x u_\beta)] - \partial_t \rho_\beta = 0. \quad (FP)_{c - \partial_x u_\beta, per}$$

Moreover, among the triples  $(\rho_\beta, u_\beta, \bar{H}_\beta(c))$  which solve  $(HJ)_{\rho_\beta, per} - (FP)_{c - \partial_x u_\beta, per}$ , there is one which minimizes

$$\int_{\mathbf{T} \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho_\beta}(t, x, \partial_x u_\beta) \rho_\beta(t, x) dt dx. \quad (2)$$

Thus, our "characteristics" are the solutions of a Fokker-Planck equation bringing mass forward in time; the drift of this equation, or the optimal trajectory, is determined by a Hamilton-Jacobi equation, backward in time. This is quite typical for this kind of problems: see for instance equation (5.40) of [17] or theorem 3.9 of [12].

We briefly sketch the proof of theorem 1; the complete details are in section 1 below. First of all, we fix  $\psi$  satisfying points d1)-d3) above; then we find, as in [13], a couple  $(u_\psi, \bar{H}_\psi(c))$  which solves  $(HJ)_{\psi,per}$ . By [13], the number  $\bar{H}_\psi(c)$  is unique and  $u_\psi$  is unique up to an additive constant. To  $c - \partial_x u_\psi$  is associated a stochastic flow, whose stationary Fokker-Planck equation is  $(FP)_{c-\partial_x u_\psi,per}$ ; again by [13],  $(FP)_{c-\partial_x u_\psi,per}$  has a unique periodic solution  $\rho_\psi$  satisfying d1)-d3). In other words, we have a map  $\psi \rightarrow \rho_\psi$  bringing densities to densities; we shall find a fixed point  $\rho_\beta$  of this map by the Schauder fixed point theorem. We shall see that  $(u_{\rho_\beta}, \rho_\beta, \bar{H}_{\rho_\beta}(c))$  solves  $(HJ)_{\rho_\beta,per} - (FP)_{c-\partial_x u_{\rho_\beta},per}$  practically by definition; the existence of a minimum in (2) will follow from the fact that the fixed points of  $\rho_\beta$  are a compact set.

In section 2, we study the Hopf-Lax semigroup. We denote by  $\mathcal{M}_1(\mathbf{T}^p)$  the space of Borel probability measures on  $\mathbf{T}^p$  with the 1-Wasserstein distance (see section 1 for a definition); we shall prove the following.

**Theorem 2.** *Let  $U: \mathcal{M}_1(\mathbf{T}^p) \rightarrow \mathbf{R}$  be of the form*

$$U(\mu) = \int_{\mathbf{T}^p} f d\mu$$

*for some  $f \in C^3(\mathbf{T}^p)$ . Let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and let  $m \in \mathbf{N}$ . Then, there are  $R_\beta \in C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$  with density  $\rho_\beta \in C^1((-m, 0] \times \mathbf{T}^p) \cap C((-m, 0] \times C^2(\mathbf{T}^p))$  and  $u_\beta \in C^1([-m, 0], C^1(\mathbf{T}^p)) \cap C([-m, 0], C^3(\mathbf{T}^p))$  such that  $u_\beta$  solves*

$$\begin{cases} \frac{1}{2\beta} \Delta u_\beta + \partial_t u_\beta - H_{\rho_\beta}(t, x, c - \partial_x u_\beta) = 0, & t < 0 \\ u_\beta(0, x) = f \quad \forall x \in \mathbf{T}^p \end{cases} \quad (HJ)_{\rho_\beta, f}$$

*and  $R_\beta$  together with its density  $\rho_\beta$  solve*

$$\begin{cases} \frac{1}{2\beta} \Delta \rho_\beta - \operatorname{div}[\rho_\beta \cdot (c - \partial_x u_\beta)] - \partial_t \rho_\beta = 0, & t > -m \\ R_\beta(-m) = \mu. \end{cases} \quad (FP)_{-m, c-\partial_x u_\beta, \mu}$$

*Among the solutions  $(u_\beta, \rho_\beta)$  of  $(HJ)_{\beta, f} - (FP)_{-m, c-\partial_x u_\beta, \mu}$ , there is one which minimizes*

$$\int_{-m}^0 dt \int_{\mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho_\beta}(t, x, c - \partial_x u_\beta) \rho_\beta dx + U(\rho_\beta(0)).$$

*We call such a minimum  $(\Lambda_c^m U)(\mu)$ .*

Since minimizing over fixed points is uncomfortable, one could ask whether this restriction can be removed, getting a problem more similar to (1).

**Theorem 3.** *1) Let  $U$  and  $(\Lambda_c^m U)(\mu)$  be as in theorem 2. Then,*

$$(\Lambda_c^m U)(\mu) = \min_Y E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\rho}(t, X, Y) dt \right\} + U(\rho(0)) \quad (3)$$

where  $\rho$  solves  $(FP)_{-m,Y,\mu}$ ,  $E_w$  denotes expectation with respect to the Brownian motion  $w$ ,  $X$  solves the stochastic differential equation

$$\begin{cases} dX(-m, s, x) = Y(s, X(-m, s, x))dt + dw(s) & s \geq -m \\ X(-m, -m, x) = X_\mu \end{cases} \quad (SDE)_{-m,Y,\mu}$$

for a random variable  $X_\mu$  of law  $\mu$ , independent on  $w(s)$  for  $s \geq -m$ . The minimum is taken over all Lipschitz vector fields  $Y$ .

2) Any minimal  $Y$  satisfies  $Y = c - \partial_x u$ , where  $u$  solves  $(HJ)_{\rho,f}$ .

In other words,  $(HJ)_{\rho_\beta,f} - (FP)_{-m,c-\partial_x u_\beta,\mu}$  are the Euler-Lagrange equations of the functional (3), exactly as in the zero-viscosity situation (we refer again to [12], theorem 3.9.) We also note a quirk of the notation: in the Hamilton-Jacobi equation we have  $H_{\rho_\beta}$ , while in (3) we have  $\mathcal{L}_{c,\frac{1}{2}\rho_\beta}$ ; again, we share this factor two with the zero-viscosity situation and we shall see the reason for it in the proof of lemma 2.5 below.

Theorems like theorem 3 are common in the theory of mean field games (see for instance [4]). In the language of mean field games, we are saying that each particle tries to minimize unilaterally the cost

$$\min_Y E_w \left\{ \int_{-m}^0 \mathcal{L}_{c,\rho}(t, X, Y) dt + f(X(0)) \right\}$$

where  $X$  solves  $(SDE)_{-m,Y,\delta_{x_0}}$  and  $\rho$  is the distribution of the other particles; this is the reason of equation  $(HJ)_{\rho_\beta,f}$  in theorem 2. The result of the independent efforts of all the particles (or the Nash equilibrium, as it is called) is that the whole community minimizes (3).

Let  $U: \mathcal{M}_1(\mathbf{T}^p) \rightarrow \mathbf{R}$  be bounded; theorem 3 prompts us to define

$$(\Psi_c^m U)(\mu) = \inf_Y \left\{ E_w \int_{-m}^0 \mathcal{L}_{c,\frac{1}{2}\rho}(t, X, Y) dt + U(\rho(0)) \right\} \quad (4)$$

where the infimum is taken over all Lipschitz vector fields  $Y$ ; the density  $\rho$  satisfies  $(FP)_{-m,Y,\mu}$ . Naturally, if  $U$  is linear as in theorem 3, then  $\Psi_c^m U = \Lambda_c^m U$ .

We shall see in proposition 2.10 below that  $\Psi_c^m$  has the semigroup property:  $\Psi_c^{m+n} = \Psi_c^m \circ \Psi_c^n$ .

Theorem 3 tells us that the infimum in (4) is a minimum when  $U$  is a linear function on measures as in theorem 2; we don't know whether this is true when  $U$  is in some more reasonable class, for instance continuous or Lipschitz. We don't even know whether, for  $U$  continuous,  $\Psi_c^1 U$  is continuous; however, when  $U$  is linear as in theorem 2, we can prove that  $\Psi_c^m U$  is Lipschitz, uniformly in  $m$ . This allows us to find, for a suitable  $\lambda \in \mathbf{R}$ , Lipschitz fixed points of the operator  $\Psi_{c,\lambda}$  defined by

$$\Psi_{c,\lambda}: U \rightarrow \Psi_c^1 U + \lambda.$$

**Theorem 4.** *There is a unique  $\lambda \in \mathbf{R}$  for which  $\Psi_{c,\lambda}$  has a fixed point  $\hat{U}$  in  $C(\mathcal{M}_1(\mathbf{T}^p), \mathbf{R})$ . In other words, for any  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ , there is a Lipschitz vector field  $\bar{Y}$  such that*

$$(\Psi_{c,\lambda}^1 \hat{U})(\mu) = E_w \left\{ \int_{-1}^0 \mathcal{L}_{c,\frac{1}{2}\bar{\rho}}(t, \bar{X}, \bar{Y}) dt \right\} + \hat{U}(\bar{\rho}(0)), \quad (5)$$

where  $X$  solves  $(SDE)_{-1, \bar{Y}, \mu}$  and  $\bar{\rho}$  solves  $(FP)_{-1, \bar{Y}, \mu}$ .

The function  $\hat{U}$  is Lipschitz for the 1-Wasserstein distance; by (5), the infimum in the definition of (4) of  $\Psi_{c, \lambda}^1 \hat{U}$  is a minimum.

The proof of this theorem is similar to the corresponding statement in Aubry-Mather theory. Indeed, using an approximation with finitely many particles, we shall prove that, for a suitable  $\lambda \in \mathbf{R}$ , the sequence  $(\Lambda_{c, \lambda})^n(0)$  of continuous functions on the compact space  $\mathcal{M}_1(\mathbf{T}^p)$  is equibounded and equilipschitz; by Ascoli-Arzelà, it has a subsequence converging to a limit  $\hat{U}$ ; we shall prove that  $\hat{U}$  is a fixed point of  $\Lambda_{c, \lambda}$ .

## §1

### Periodic orbits

In this section, we are going to prove theorem 1. We begin with a study of  $(HJ)_{\psi, per}$ ; we follow the approach of [13] but, for completeness' sake, we reprove several results of this paper using, as in [2], the Feynman-Kac formula.

#### Definitions.

- We group in a set  $Den$  the functions on  $\mathbf{T} \times \mathbf{T}^p$  which satisfy points d1)-d3) in the introduction. Clearly, the set  $Den$  is closed in  $C(\mathbf{T} \times \mathbf{T}^p)$ .
- We define  $\mathcal{M}_1(\mathbf{T}^p)$  as the space of all Borel probability measures on  $\mathbf{T}^p$ ; if  $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbf{T}^p)$ , we define the 1-Wasserstein distance between them as

$$d_1(\mu_1, \mu_2) = \min \left\{ \int_{\mathbf{T}^p \times \mathbf{T}^p} |x - x'|_{\mathbf{T}^p} d\gamma(x, x') \right\}$$

where  $|x - x'|_{\mathbf{T}^p}$  is the distance on the flat torus  $\mathbf{T}^p$ . The minimum is taken over all the Borel probability measures  $\gamma$  on  $\mathbf{T}^p \times \mathbf{T}^p$  whose first and second marginals are, respectively,  $\mu_1$  and  $\mu_2$ . It is standard (see for instance section 7.1 of [17]) that  $d_1$  turns  $\mathcal{M}_1(\mathbf{T}^p)$  into a complete metric space, and induces the weak\* topology.

We note that, if  $\psi \in Den$  and  $\mathcal{L}^p$  denotes the Lebesgue measure on  $\mathbf{T}^p$ , then the function  $: t \rightarrow \psi(t, \cdot) \mathcal{L}^p$  belongs to  $C(\mathbf{T}, \mathcal{M}_1(\mathbf{T}^p))$ .

- We extend the definition of  $P_\psi$  we gave in the introduction: for  $\psi \in C(\mathbf{R}, \mathcal{M}_1(\mathbf{T}^p))$  we set

$$P_\psi(t, x) = V(t, x) + \int_{\mathbf{T}^p} W(x - x') d\psi(t, x') \quad (1.1)$$

**Lemma 1.1.** *There is  $C_1 > 0$ , independent on  $\psi \in C(\mathbf{R}, \mathcal{M}_1(\mathbf{T}^p))$ , such that the function  $P_\psi(t, x)$  defined in (1.1) satisfies*

$$\|P_\psi\|_{C(\mathbf{R}, C^3(\mathbf{T}^p))} \leq C_1. \quad (1.2)$$

**Proof.** We recall that, by definition,

$$\|P_\psi\|_{C(\mathbf{R}, C^3(\mathbf{T}^p))} = \sup_{t \in \mathbf{R}} \|P_\psi(t, \cdot)\|_{C^3(\mathbf{T}^p)}$$

where, as usual,

$$\|f\|_{C^3(\mathbf{T}^p)} = \|f\|_{C^0(\mathbf{T}^p)} + \|D_x f\|_{C^0(\mathbf{T}^p)} + \|D_x^2 f\|_{C^0(\mathbf{T}^p)} + \|D_x^3 f\|_{C^0(\mathbf{T}^p)}.$$

By our hypotheses on  $V$  and  $W$ , we have that

$$\|V\|_{C(\mathbf{R}, C^3(\mathbf{T}^p))} + \|W\|_{C^3(\mathbf{T}^p)} = C_1 < +\infty. \quad (1.3)$$

For  $0 \leq j \leq 3$ , differentiation under the integral sign implies that

$$D_x^j P_\psi(t, x) = D_x^j V(t, x) + \int_{\mathbf{T}^p} D_x^j W(x - x') d\psi(t, x').$$

Since  $t \rightarrow \psi(t, \cdot)$  is continuous from  $\mathbf{R}$  to the weak\* topology, the formula above implies that  $P_\psi$  is in  $C(\mathbf{R}, C^3(\mathbf{T}^p))$ . Since  $\psi(t, \cdot)$  is a probability measure and the  $C^3$  norm is convex, (1.2) follows from the last formula and (1.3).

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From now on, we shall fix  $\psi \in Den$ ; the functions  $P_\psi$ ,  $\mathcal{L}_{c, \psi}$  and  $H_\psi$  are defined as in the introduction. Following [2], we note that, if  $(u, A)$  solves

$$\begin{cases} \frac{1}{2\beta} \Delta u + \partial_t u - H_\psi(t, x, c - \partial_x u) + A = 0 & \forall t \in \mathbf{R} \\ u(t, \cdot) = u(t+1, \cdot) & \forall t \in \mathbf{R} \end{cases} \quad (HJ)_{\psi, per}$$

and is periodic in space (i. e. it quotients to a continuous function on  $\mathbf{T}^p$ ), then the couple  $(v, A) = (e^{-\beta u}, A)$  is a solution, periodic in space, of the "twisted" Schrödinger equation

$$\begin{cases} \partial_t v + e^{-\beta \langle c, x \rangle} \left[ \frac{1}{2\beta} \Delta + \beta P_\psi(t, x) - \beta A \right] (e^{\beta \langle c, x \rangle} v) = 0 & \forall t \in \mathbf{R} \\ v(t, \cdot) = v(t+1, \cdot) & \forall t \in \mathbf{R}. \end{cases} \quad (TS)_{\psi, per}$$

Vice-versa, the logarithm of a positive solution of  $(TS)_{\psi, per}$  solves  $(HJ)_{\psi, per}$ . Thus, solving  $(HJ)_{\psi, per}$  reduces to solving  $(TS)_{\psi, per}$ ; that's what we are going to do next.

For  $\phi \in C(\mathbf{T}^p)$  and  $A \in \mathbf{R}$  we consider the evolution (or involution, since it goes backward in time) problem

$$\begin{cases} \partial_t v + e^{-\beta \langle c, x \rangle} \left[ \frac{1}{2\beta} \Delta + \beta P_\psi(t, x) - \beta A \right] (e^{\beta \langle c, x \rangle} v) = 0, & t \leq 0 \\ v(0, x) = \phi(x). \end{cases} \quad (TS)_{\psi, \phi}$$

If  $t \leq 0$ , we can use the Feynman-Kac formula (see for instance [6]) and write the unique solution of  $(TS)_{\psi, \phi}$  as

$$v(t, x) = (L_{(\psi, A, t)} \phi)(x), \quad t \leq 0$$

where

$$(L_{(\psi,A,t)}\phi)(x) = e^{-\beta\langle c,x \rangle} \cdot E_w \left[ e^{\int_t^0 \beta [P_\psi(\tau, \frac{1}{\sqrt{\beta}} w(\tau) + x) - A] d\tau} e^{\beta\langle c, \frac{1}{\sqrt{\beta}} w(0) + x \rangle} \phi \left( \frac{1}{\sqrt{\beta}} w(0) + x \right) \right]. \quad (1.5)$$

In the formula above,  $w$  is a Brownian motion on  $[t, +\infty]$  with  $w(t) = 0$ , and  $E_w$  is the expectation with respect to the Wiener measure.

We shall see in lemma 1.4 below that there is a bijection between the positive eigenfunctions of  $L_{(\psi,0,-1)}$  and the positive solutions of  $(TS)_{\beta,per}$ ; now, we prove that such eigenfunctions exist.

**Lemma 1.2.** 1) (Existence) there is  $(v, B) \in C(\mathbf{T}^p) \times \mathbf{R}$  such that

$$\begin{cases} L_{(\psi,0,-1)}v = Bv \\ v > 0 \\ B > 0. \end{cases} \quad (1.6)_\psi$$

2) (Uniqueness) Let  $(v_1, B_1)$  and  $(v_2, B_2)$  solve  $(1.6)_\psi$ ; then,  $B_1 = B_2$  and  $v_1 = \alpha v_2$  for some  $\alpha > 0$ . In particular, there is a unique couple  $(v_\psi, B_\psi)$  which satisfies  $(1.6)_\psi$  and such that

$$\int_{\mathbf{T}^p} v_\psi(x) dx = 1. \quad (1.7)$$

**Proof.** We recall from [16] (see also chapter XVI of [3] for G. Birkhoff's original exposition) a few facts about the Perron-Frobenius theorem. Let us denote by  $\mathcal{C}_+ \subset C(\mathbf{T}^p)$  the cone of strictly positive, continuous functions. We forego the easy proof that  $L_{(\psi,0,-1)}$  brings  $\mathcal{C}_+$  into itself.

Let  $v_1, v_2 \in \mathcal{C}_+$ ; we say that  $v_1$  and  $v_2$  are equivalent, or  $v_1 \simeq v_2$ , if  $v_1 = tv_2$  for some  $t > 0$ . Given  $v_1, v_2 \in \mathcal{C}_+$ , we define

$$\alpha(v_1, v_2) = \sup\{t > 0 : v_2 - tv_1 \in \mathcal{C}_+\}$$

and

$$\theta(v_1, v_2) = -\log[\alpha(v_1, v_2)\alpha(v_2, v_1)].$$

It turns out ([16]) that  $(\frac{\mathcal{C}_+}{\simeq}, \theta)$  is a complete metric space. We refer again to [16] or [3] for the proof that

$$\theta(L_{(\psi,0,-1)}v_1, L_{(\psi,0,-1)}v_2) \leq (1 - e^{-D})\theta(v_1, v_2)$$

where

$$D = \sup_{v_1, v_2 \in \mathcal{C}_+} \theta(L_{(\psi,0,-1)}v_1, L_{(\psi,0,-1)}v_2).$$

As a consequence, points 1) and 2) follow from the contraction mapping theorem if we prove that  $D < +\infty$ . Actually, we are going to show that  $D$  is bounded from above independently of  $\psi \in Den$ ; equivalently, the Lipschitz constant of  $L_{(\psi,0,-1)}$  does not depend on  $\psi$ . We shall need this fact in the next lemma.

Let  $v_1, v_2 \in \mathcal{C}_+$ . Recalling the definition of  $\theta$ , we see that

$$\theta(v_1, v_2) \leq \log \left( \frac{\max v_2}{\min v_1} \cdot \frac{\max v_1}{\min v_2} \right). \quad (1.8)$$

Thus,  $D < +\infty$  follows if we prove that there is  $C_3 > 0$  such that

$$\frac{\max L_{(\psi,0,-1)}v}{\min L_{(\psi,0,-1)}v} \leq C_3 \quad (1.9)$$

for all  $v \in \mathcal{C}_+$ ; since the term on the left is homogeneous of degree zero in  $v$ , we can suppose that  $v$  satisfies (1.7).

We prove (1.9); in the following,  $C_i$  always denotes a constant independent on  $v$  and  $\psi$ . By (1.5) and the fact that  $v > 0$ , we have that

$$(L_{(\psi,0,-1)}v)(x) \geq e^{-\beta\langle c,x \rangle} e^{\beta \min P_\psi} \frac{1}{\sqrt{(2\pi)^p}} \int_{\mathbf{R}^p} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v\left(\frac{1}{\sqrt{\beta}}z+x\right) e^{-\frac{|z|^2}{2}} dz.$$

Setting  $\frac{1}{\sqrt{\beta}}z = y$  and simplifying  $e^{-\beta\langle c,x \rangle}$  outside the integral with  $e^{\beta\langle c,x \rangle}$  inside, we get the first inequality below

$$\begin{aligned} (L_{(\psi,0,-1)}v)(x) &\geq e^{\beta \min P_\psi} \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \int_{\mathbf{R}^p} e^{\beta\langle c,y \rangle} v(x+y) e^{-\frac{\beta}{2}|y|^2} dy \geq \\ &e^{-\beta C_1} \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \int_{[0,1]^p} e^{\beta\langle c,y \rangle} v(x+y) e^{-\frac{\beta}{2}|y|^2} dy. \end{aligned}$$

The second inequality above comes from lemma 1.1 and the fact that  $v$ , which belongs to  $\mathcal{C}_+$ , is positive. By lemma 1.1, the constant  $C_1$  does not depend on  $\psi \in Den$ .

We assert that

$$(L_{(\psi,0,-1)}v)(x) \geq e^{-\beta C_1} \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \min_{y \in [0,1]^p} \left[ e^{\beta\langle c,y \rangle} e^{-\frac{\beta}{2}|y|^2} \right] \int_{[0,1]^p} v(x+y) dy = C_5 \quad (1.10)$$

for a constant  $C_5 > 0$  independent on  $\psi$  and  $v$ . Indeed, the inequality follows since  $v$  is positive; since  $v$  is periodic and satisfies (1.7), the integral above is 1, and the equality follows.

For the estimate from above, we get from (1.5) that

$$(L_{(\psi,0,-1)}v)(x) \leq e^{-\beta\langle c,x \rangle} e^{\beta \max P_\psi} \frac{1}{\sqrt{(2\pi)^p}} \int_{\mathbf{R}^p} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v\left(\frac{1}{\sqrt{\beta}}z+x\right) e^{-\frac{|z|^2}{2}} dz.$$

We simplify  $e^{-\beta\langle c,x \rangle}$  outside the integral with  $e^{\beta\langle c,x \rangle}$  inside; now lemma 1.1 gives us the first inequality below; the equality follows from the change of variables  $\frac{1}{\sqrt{\beta}}z = y$ .

$$\begin{aligned} (L_{(\psi,0,-1)}v)(x) &\leq \frac{e^{\beta C_1}}{\sqrt{(2\pi)^p}} \int_{\mathbf{R}^p} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z \rangle} v\left(\frac{1}{\sqrt{\beta}}z+x\right) e^{-\frac{|z|^2}{2}} dz = \\ &e^{\beta C_1} \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \int_{\mathbf{R}^p} e^{\beta\langle c,y \rangle} v(x+y) e^{-\frac{\beta}{2}|y|^2} dy. \end{aligned}$$

Since  $v$  is positive periodic, and by (1.7) integrates to 1 on the unit cube, we get the first inequality below.

$$(L_{(\psi,0,-1)}v)(x) \leq e^{\beta C_1} \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \cdot \sum_{k \in \mathbf{Z}^p} \max_{y \in k+[0,1]^p} \left[ e^{\beta\langle c,y \rangle} e^{-\frac{\beta}{2}|y|^2} \right] \leq$$



$$C_6 \sum_{k \in \mathbf{Z}^p} e^{\beta|c|(|k|+\sqrt{p})} e^{-\frac{\beta}{2}(|k|-\sqrt{p})^2}.$$

Since the sum in the last formula is finite, we get that

$$(L_{(\psi,0,-1)}v)(x) \leq C_7 \quad \forall x \in \mathbf{T}^p \quad (1.11)$$

for a constant  $C_7 > 0$  independent on  $\psi$  and  $v$ . Now (1.9) follows from the last formula and (1.10); we have seen that (1.9), by the contraction mapping theorem, implies points 1) and 2) of the thesis.

\\

**Lemma 1.3.** *Let  $\psi \in Den$  and let  $(v_\psi, B_\psi)$  be as in the last lemma. Then,  $v_\psi \in C^3(\mathbf{T}^p)$  and the following two points hold.*

1) (Uniform estimates) *There is  $C_8 > 0$ , independent on  $\psi \in Den$ , such that*

$$i) \quad \|v_\psi\|_{C^3(\mathbf{T}^p)} \leq C_8,$$

$$ii) \quad \frac{1}{C_8} \leq v_\psi(x) \leq C_8 \quad \forall x \in \mathbf{T}^p$$

$$iii) \quad \frac{1}{C_8} \leq B_\psi \leq C_8.$$

2) (Continuous dependence) *The function*

$$K: Den \rightarrow C^3(\mathbf{T}^p) \times \mathbf{R}, \quad K: \psi \rightarrow (v_\psi, B_\psi)$$

*is continuous.*

**Proof.** We prove point 1). Since  $v_\psi$  satisfies (1.7), by (1.10) and (1.11) there is  $C_8 > 1$  such that

$$\frac{1}{C_8} \leq \min L_{(\psi,0,-1)}v_\psi \leq \max L_{(\psi,0,-1)}v_\psi \leq C_8.$$

Since we also have that  $L_{(\psi,0,-1)}v_\psi = B_\psi v_\psi$ , we get that

$$\frac{1}{B_\psi} \cdot \frac{1}{C_8} \leq \min v_\psi \leq v_\psi \leq \max v_\psi \leq \frac{1}{B_\psi} C_8. \quad (1.12)$$

Integrating on  $\mathbf{T}^p$  and using (1.7), we get that

$$\frac{1}{B_\psi} \cdot \frac{1}{C_8} \leq \int_{\mathbf{T}^p} v_\psi dx = 1 \leq \frac{1}{B_\psi} \cdot C_8$$

from which *iii)* of point 1) follows.

From point *iii)* and (1.12), possibly increasing  $C_8$ , we get point *ii)*. We show *i)*.

We would like to differentiate under the integral sign in (1.5); we cannot do this immediately, because we only know that the final condition  $\phi$  (which in our case is  $v_\psi$ ) is in  $C^0$ . Let  $E_{(0,z)}$  denote the expectation of the Brownian bridge with  $w(-1) = 0$  and  $w(0) = z$ ; by (1.5) we get that, for  $v \in \mathcal{C}_+$ ,

$$(L_{(\psi,0,-1)}v)(x) = e^{-\beta\langle c,x \rangle} \frac{1}{\sqrt{(2\pi)^p}} \cdot \int_{\mathbf{R}^p} e^{-\frac{|z|^2}{2}} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v\left(\frac{1}{\sqrt{\beta}}z+x\right) \cdot E_{(0,z)} \left[ e^{\int_{-1}^0 \beta P_\psi(\tau, \frac{1}{\sqrt{\beta}}w(\tau)+x) d\tau} \right] dz.$$

Setting  $\frac{1}{\sqrt{\beta}}z+x = y$ , we get that

$$(L_{(\psi,0,-1)}v)(x) = \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \cdot e^{-\beta\langle c,x \rangle} \cdot \int_{\mathbf{R}^p} e^{-\frac{\beta}{2}|y-x|^2} e^{\beta\langle c,y \rangle} v(y) \cdot E_{(0,\sqrt{\beta}(y-x))} \left[ e^{\int_{-1}^0 \beta P_\psi(\tau, \frac{1}{\sqrt{\beta}}w(\tau)+x) d\tau} \right] dy.$$

We recall from [14] that, if  $\tilde{w}$  is a Brownian bridge with  $\tilde{w}(-1) = \tilde{w}(0) = 0$ , then  $w(t) := \sqrt{\beta}(y-x)(t+1) + \tilde{w}(t)$  is a Brownian bridge with  $w(-1) = 0$ ,  $w(0) = \sqrt{\beta}(y-x)$ . This and the last formula imply that

$$(L_{(\psi,0,-1)}v)(x) = \sqrt{\left(\frac{\beta}{2\pi}\right)^p} \cdot e^{-\beta\langle c,x \rangle} \cdot \int_{\mathbf{R}^p} e^{-\frac{\beta}{2}|y-x|^2} e^{\beta\langle c,y \rangle} v(y) \cdot E_{(0,0)} \left[ e^{\int_{-1}^0 \beta P_\psi(\tau, (y-x)(\tau+1) + \frac{1}{\sqrt{\beta}}\tilde{w}(\tau)+x) d\tau} \right] dy.$$

The formula above allows us to differentiate under the integral sign, even if  $v$  is only  $C^0$ ; using lemma 1.1, we easily get

$$\|L_{(\psi,0,-1)}v\|_{C^3(\mathbf{T}^p)} \leq C_9 \|v\|_{C^0(\mathbf{T}^p)} \quad (1.13)$$

for a constant  $C_9$  independent of  $\psi$ . By *ii*), we get that

$$\|L_{(\psi,0,-1)}v\|_{C^3(\mathbf{T}^p)} \leq C_8 \cdot C_9.$$

Since  $L_{(\psi,0,-1)}v_\psi = B_\psi v_\psi$ , formula *i*) now follows from *iii*).

We prove point 2); in the first three steps below, we show a weaker result, namely that the map  $:\psi \rightarrow v_\psi$  is continuous from  $Den$  to  $C^0(\mathbf{T}^p)$ ; this will follow from the theorem of contractions depending on a parameter applied to the map

$$\Xi: (Den, \|\cdot\|_{\sup}) \times (\mathcal{C}_+, \theta) \rightarrow (\mathcal{C}_+, \theta), \quad \Xi: (\psi, v) \rightarrow L_{(\psi,0,-1)}v.$$

**Step 1.** We begin to observe that  $\theta$  and the sup norm induce equivalent topologies on the subset  $\mathcal{A}$  of the functions of  $\mathcal{C}_+$  which satisfy (1.7). Indeed, (1.8) proves that the  $C^0$  topology is stronger; for the opposite inclusion, let  $\theta(v_n, v) \rightarrow 0$  and let  $v_n, v$  satisfy (1.7). Since  $\theta(v_n, v) \rightarrow 0$ , we have that, for any  $\epsilon > 0$  and  $n$  large enough,

$$\frac{1-\epsilon}{\alpha(v_n, v)} \leq \alpha(v, v_n) \leq \frac{1+\epsilon}{\alpha(v_n, v)}.$$

The definition of  $\alpha$  implies the first two inequalities below; the last one follows by the first inequality above.

$$\alpha(v, v_n)v \leq v_n \leq \frac{1}{\alpha(v_n, v)}v \leq \frac{\alpha(v, v_n)}{1 - \epsilon}v.$$

Since  $v$  and  $v_n$  satisfy (1.7), if we integrate the formula above on  $\mathbf{T}^p$ , we get that  $\alpha(v_n, v) \rightarrow 1$  and that  $\alpha(v, v_n) \rightarrow 1$ ; since  $\min v > 0$ , again from the formula above we get that  $v_n \rightarrow v$  uniformly.

**Step 2.** Let  $v \in \mathcal{C}_+$  be fixed; we assert that the map  $:\psi \rightarrow \Xi(\psi, v)$  is continuous from the  $\|\cdot\|_{\text{sup}}$  to the  $\theta$  topology. Indeed, we saw in step 1 that, on  $\mathcal{C}_+$ , the  $C^0$  topology is stronger than the  $\theta$  topology; thus, it suffices to prove that  $\Xi(\cdot, v): (Den, \|\cdot\|_{\text{sup}}) \rightarrow (\mathcal{C}_+, \|\cdot\|_{\text{sup}})$  is continuous. The proof of this, which ends the proof of the assertion, follows by applying the theorem of continuity under the integral sign to (1.5), and we forego it.

**Step 3.** We assert that the map  $:\psi \rightarrow v_\psi$  is continuous from  $(Den, \|\cdot\|_{\text{sup}})$  to  $(\mathcal{A}, \|\cdot\|_{\text{sup}})$ ; by step 1, it suffices to prove that it is continuous from  $(Den, \|\cdot\|_{\text{sup}})$  to  $(\mathcal{A}, \theta)$ . We have seen in the proof of lemma 1.2 that  $:\psi \rightarrow \Xi(\psi, v)$  is a contraction for the  $\theta$ -topology, whose Lipschitz constant does not depend on  $\psi$ . Since  $:\psi \rightarrow \Xi(\psi, v)$  is continuous by step 2, we can apply the theorem of contractions depending on a parameter, and get that the map  $:\psi \rightarrow v_\psi$  is continuous from  $(Den, \|\cdot\|_{\text{sup}})$  to  $(\mathcal{C}_+, \theta)$ , as we wanted.

**Step 4.** We assert that the map  $:\psi \rightarrow B_\psi$  is continuous from  $Den$  to  $\mathbf{R}$ . Since  $L_{(\psi, 0, -1)}v_\psi = B_\psi v_\psi$ , it suffices to prove that both maps  $:\psi \rightarrow v_\psi$  and  $:\psi \rightarrow L_{(\psi, 0, -1)}v_\psi$  are continuous from  $Den$  to  $C^0(\mathbf{T}^p)$ . The first fact has been proven in step 3; we prove that  $:\psi \rightarrow L_{(\psi, 0, -1)}v_\psi$  is continuous. Indeed,

$$\|L_{(\psi', 0, -1)}v_{\psi'} - L_{(\psi, 0, -1)}v_\psi\|_{\text{sup}} \leq \|L_{(\psi', 0, -1)}(v_{\psi'} - v_\psi)\|_{\text{sup}} + \|(L_{(\psi', 0, -1)} - L_{(\psi, 0, -1)})v_\psi\|_{\text{sup}}.$$

Now the assertion follows from the fact that (with the sup norm in all spaces)  $:\psi \rightarrow v_\psi$  is continuous, that  $:\psi \rightarrow L_{(\psi, 0, -1)}v$  is continuous, and that  $:\psi \rightarrow L_{(\psi, 0, -1)}v$  is uniformly Lipschitz by (1.13).

**End of the proof of point 2).** For  $\phi \in \mathcal{C}_+$ , we get from (1.5) that

$$L_{(\psi, r, -1)}\phi = e^{-\beta r}L_{(\psi, 0, -1)}\phi. \quad (1.14)$$

Setting  $A_\psi = \frac{1}{\beta} \log B_\psi$ , the formula above implies that

$$L_{(\psi, A_\psi, -1)}v_\psi = v_\psi. \quad (1.15)$$

The same proof which yielded (1.13) also yields that there is  $C_{10} > 0$  such that, if  $A$  and  $A'$  satisfy the estimate of point 1), *iii*) of this lemma, then

$$\|L_{(\psi, A, -1)}v - L_{(\psi', A', -1)}v\|_{C^3(\mathbf{T}^p)} \leq C_{10}(\|\psi - \psi'\|_{C^0(\mathbf{T}^p)} + |A - A'|) \cdot \|v\|_{C^0(\mathbf{T}^p)}. \quad (1.16)$$

Thus,

$$\begin{aligned} \|v_\psi - v_{\psi'}\|_{C^3(\mathbf{T}^p)} &= \|L_{(\psi, A_\psi, -1)}v_\psi - L_{(\psi', A_{\psi'}, -1)}v_{\psi'}\|_{C^3(\mathbf{T}^p)} \leq \\ &\|L_{(\psi, A_\psi, -1)}(v_\psi - v_{\psi'})\|_{C^3(\mathbf{T}^p)} + \|(L_{(\psi, A_\psi, -1)} - L_{(\psi', A_{\psi'}, -1)})v_{\psi'}\|_{C^3(\mathbf{T}^p)} \leq \\ &C_9\|v_\psi - v_{\psi'}\|_{C^0(\mathbf{T}^p)} + C_{10}(\|\psi - \psi'\|_{C^0(\mathbf{T}^p)} + |A_\psi - A_{\psi'}|) \cdot \|v_{\psi'}\|_{C^0(\mathbf{T}^p)} \end{aligned}$$

where the equality comes from (1.15) and the last inequality comes from (1.13) and (1.16). Since the map  $\psi \rightarrow v_\psi$  is continuous from  $Den$  to the  $C^0$  topology by step 3, and  $\psi \rightarrow A_\psi$  is continuous too (because  $\psi \rightarrow B_\psi$  is continuous by step 4 and point 1), *iii*) of this lemma holds), point 2) follows.

\\

In the next lemma, we show how the fixed points of  $L_{(\psi,0,-1)}$  induce solutions of  $(TS)_{\psi,per}$ .

**Lemma 1.4.** 1) (Existence) Given  $\psi \in Den$ , we can find  $A \in \mathbf{R}$  and  $\hat{v} \in C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))$  such that  $\hat{v} > 0$  and  $(\hat{v}, A)$  solves  $(TS)_{\psi,per}$ .

2) (Uniqueness) Let us suppose that  $(\hat{v}, A)$  and  $(\hat{v}^1, A^1)$  are two solutions of  $(TS)_{\psi,per}$  with  $\hat{v} > 0$  and  $\hat{v}^1 > 0$ . Then,  $A = A^1$  and  $\hat{v} = \lambda \hat{v}^1$  for some  $\lambda > 0$ .

3) (Estimates) Let us call  $(\hat{v}_\psi, A_\psi)$  the solution of  $(TS)_{\psi,per}$  such that  $\hat{v}_\psi > 0$  and  $\hat{v}_\psi(0, \cdot)$  satisfies (1.7). Then, there is a constant  $C_{10} > 0$ , independent on  $\psi \in Den$ , such that

$$|A_\psi| + \|\hat{v}_\psi\|_{C(\mathbf{T}, C^3(\mathbf{T}^p))} + \|\hat{v}_\psi\|_{C^1(\mathbf{T}, C^1(\mathbf{T}^p))} \leq C_{10} \quad (1.17)$$

and

$$\frac{1}{C_{10}} \leq \hat{v}_\psi(t, x) \leq C_{10} \quad \forall (t, x) \in \mathbf{T} \times \mathbf{T}^p. \quad (1.18)$$

4) (Continuous dependence) Let  $(\hat{v}_\psi, A_\psi)$  be as in point 3), and let us consider the map  $I: \psi \rightarrow (\hat{v}_\psi, A_\psi)$ . Then,  $I$  is continuous from  $Den$  to  $[C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))] \times \mathbf{R}$ .

**Proof.** As in the proof of lemma 1.3, we set  $A_\psi = \frac{1}{\beta} \log B_\psi$ . For  $t \leq 0$ , we set

$$\hat{v}_\psi(t, x) = (L_{(\psi, A_\psi, t)} v_\psi)(x). \quad (1.19)$$

By (1.15), we get that  $\hat{v}_\psi(-1, x) = \hat{v}(0, x)$ ; in other words,  $\hat{v}_\psi$  quotients on  $\mathbf{T} \times \mathbf{T}^p$ ; equivalently, it satisfies the second formula of  $(TS)_{\psi,per}$ .

Let us prove (1.18) for the function  $\hat{v}_\psi$  defined by (1.19); we prove the inequality on the left, since the one on the right is analogous.

The first equality below is (1.19). Since  $\hat{v}_\psi$  is periodic, we can suppose that  $t \in [-1, 0]$ ; now (1.5), implies the first inequality below; the second inequality follows from lemma 1.1 and the fact that  $t \in [-1, 0]$ ; the third one comes from point 1), *ii*) and *iii*) of lemma 1.3.

$$\begin{aligned} \hat{v}_\psi(t, x) &= (L_{(\psi, A_\psi, t)} v_\psi)(x) \geq e^{-A_\psi} e^{\min(t\beta P_\psi)} (\min v_\psi) E_w \left( e^{\beta \langle c, \sqrt{\beta} w(0) \rangle} \right) = \\ &e^{-A_\psi} e^{\min(t\beta P_\psi)} (\min v_\psi) \frac{1}{\sqrt{(2\pi|t|)^p}} \int_{\mathbf{R}^p} e^{-\frac{|x|^2}{2|t|}} e^{\sqrt{\beta} \langle c, x \rangle} dx \geq \\ &e^{-A_\psi} C_9 \min v_\psi \geq \frac{C_9}{C_8}. \end{aligned}$$

This yields the inequality on the left of (1.18).

We prove (1.17). We begin to note that the estimate on  $A_\psi$  follows by point 1), *iii*) of lemma 1.3, and by the fact that  $A_\psi = \frac{1}{\beta} \log B_\psi$ .

We end the proof of (1.17) with the estimates on the derivatives. Let  $\tilde{w}$  be the Brownian bridge with  $\tilde{w}(-1) = 0 = \tilde{w}(0)$  and let  $\tilde{E}_{(0,0)}$  denote its expectation; for  $t < 0$ , let  $w$  be the Brownian bridge with  $w(t) = 0 = w(0)$  and let  $E_{(0,0)}$  denote its expectation; we recall that

$$w(s) = \frac{1}{\sqrt{|t|}} \tilde{w}\left(\frac{s}{|t|}\right).$$

This yields the second inequality below, while (1.19) yields the first one; the third one comes from the change of variables  $s = \frac{\tau}{|t|}$ .

$$\begin{aligned} \hat{v}_\psi(t, x) &= \left(\frac{\beta}{2\pi|t|}\right)^{\frac{p}{2}} e^{-\beta\langle c, x \rangle}. \\ \int_{\mathbf{R}^p} e^{-\frac{|z|^2}{2|t|}} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v_\psi\left(\frac{1}{\sqrt{\beta}}z+x\right) E_{(0,0)}\left[e^{\int_t^0 \beta P_\psi(\tau, x + \frac{1}{\sqrt{\beta}}w(\tau) + \frac{\tau+t}{|t|}z) d\tau}\right] dz &= \\ \left(\frac{\beta}{2\pi|t|}\right)^{\frac{p}{2}} e^{-\beta\langle c, x \rangle}. \\ \int_{\mathbf{R}^p} e^{-\frac{|z|^2}{2|t|}} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v_\psi\left(\frac{1}{\sqrt{\beta}}z+x\right) \tilde{E}_{(0,0)}\left[e^{\int_t^0 \beta P_\psi(\tau, x + \frac{1}{\sqrt{\beta|t|}}w(\frac{\tau}{|t|}) + \frac{\tau+t}{|t|}z) d\tau}\right] dz &= \\ \left(\frac{\beta}{2\pi|t|}\right)^{\frac{p}{2}} e^{-\beta\langle c, x \rangle}. \\ \int_{\mathbf{R}^p} e^{-\frac{|z|^2}{2|t|}} e^{\beta\langle c, \frac{1}{\sqrt{\beta}}z+x \rangle} v_\psi\left(\frac{1}{\sqrt{\beta}}z+x\right) \tilde{E}_{(0,0)}\left[e^{\int_{-1}^0 \beta P_\psi(|t|s, x + \frac{1}{\sqrt{\beta|t|}}w(s) + (s+1)z) ds}\right] dz &= \end{aligned} \quad (1.20)$$

By point 1), *i*) of lemma 1.3, we can differentiate under the integral sign and get that

$$\|\hat{v}_\psi\|_{C([-2, -1], C^3(\mathbf{T}^p))} + \|\hat{v}_\psi\|_{C^1([-2, -1], C^1(\mathbf{T}^p))} \leq C_{10}.$$

Since  $\hat{v}_\psi$  is periodic in time, (1.17) follows.

By theorem 9.1 and proposition 6.6 of [6], the Feynman-Kac formula holds for the unbounded final condition  $e^{\beta\langle c, x \rangle} v_\psi$ ; this, (1.19) and (1.5) imply that  $\hat{v}_\psi$  satisfies the first formula of  $(TS)_{\psi, per}$  for  $t < 0$ ; since it is periodic in  $t$ , it satisfies it for all times. Moreover,  $\hat{v}_\psi > 0$  because, by (1.19) and (1.5), it is an integral, with a positive weight, of the positive  $v_\psi$ . This ends the proof of point 1).

We have just seen that (1.19) gives a bijection between the periodic, positive solutions of  $(TS)_{\psi, per}$  and the positive eigenfunctions of  $L_{(\psi, 0, -1)}$ ; since the latter are unique up to a multiplicative constant by point 2) of lemma 1.2, we get that the former too are unique up to a multiplicative constant; this proves point 2).

We prove point 4). To prove that the map  $:\psi \rightarrow A_\psi$  is continuous, it suffices to note that  $A_\psi = \frac{1}{\beta} \log B_\psi$ , that the map  $:\psi \rightarrow B_\psi$  is continuous by point 2) of lemma 1.3, and that  $B_\psi$  is bounded away from zero and infinity by point 1), *iii*) of the same lemma.

By point 2) of lemma 1.3, we know that  $:\psi \rightarrow v_\psi$  is continuous from  $Den$  to  $C^3(\mathbf{T}^p)$ ; this and (1.19) easily imply that  $:\psi \rightarrow \hat{v}_\psi$  is continuous from  $Den$  to  $C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))$ .

\\

**Lemma 1.5.** 1) (Existence and uniqueness) There is a unique couple  $\hat{H}_\psi(c) \in \mathbf{R}$  and  $u_\psi \in C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))$  which solves  $(HJ)_{\psi, per}$  and satisfies

$$\int_{\mathbf{T}^p} u_\psi(0, x) dx = 0. \quad (1.21)$$

2) (Estimates) There is  $C_{11} > 0$ , independent on  $\psi \in Den$ , such that, if  $u_\psi$  is as in point 1), then

$$|\bar{H}_\psi(c)| + \|u_\psi\|_{C(\mathbf{T}, C^3(\mathbf{T}^p))} + \|u_\psi\|_{C^1(\mathbf{T}, C^1(\mathbf{T}^p))} \leq C_{11}. \quad (1.22)$$

3) (Continuous dependence) The couple  $(u_\psi, \bar{H}_\psi(c))$  depends continuously on  $\psi$ : if  $\psi_n \rightarrow \psi$  in  $Den$ , then  $\bar{H}_{\psi_n}(c) \rightarrow \bar{H}_\psi(c)$  in  $\mathbf{R}$  and  $u_{\psi_n} \rightarrow u_\psi$  in  $C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))$ .

**Proof.** By lemma 1.4, there is a unique couple

$$(\hat{v}_\psi, A_\psi) \in [C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))] \times \mathbf{R}$$

which solves  $(TS)_{\psi, per}$  and such that  $\hat{v}_\psi(0, \cdot)$  is positive and satisfies (1.7). We have seen at the beginning of this section that, for any  $\lambda > 0$ , the couple

$$(u_\psi, \bar{H}_\psi(c)) := \left(-\frac{1}{\beta} \log(\lambda \hat{v}_\psi), -\frac{1}{\beta} A_\psi\right) \quad (1.23)$$

solves  $(HJ)_{\psi, per}$ ; vice-versa, if  $u_\psi$  solves  $(HJ)_{\psi, per}$ , then its exponential solves  $(TS)_{\psi, per}$ . Thus, if we define  $u_\psi$  as above, for the unique  $\lambda$  for which (1.21) holds, we have existence. Now point 2) of lemma 1.4 implies that all positive solutions of  $(TS)_{\psi, per}$  are of the form  $(\lambda \hat{v}_\psi, A_\psi)$ ; since we have just seen that there is a bijection between the solutions of  $(HJ)_{\psi, per}$  and the positive solutions of  $(TS)_{\psi, per}$ , we get uniqueness.

Formula (1.22) follows from (1.23); indeed, the derivatives of the logarithm of  $\hat{v}_\psi$  are bounded by (1.17) and (1.18). In an analogous way, point 3) follows from point 4) of lemma 1.4.

\\

Let the Lagrangian  $\mathcal{L}_{c, \psi}$  be as in the introduction, and let  $u_\psi$  be as in lemma 1.5. It is well-known ([9]) that  $u_\psi$  satisfies, for  $t \leq 0$ ,

$$u_\psi(t, x) = \min_Y E_w \left\{ \int_t^0 \mathcal{L}_{c, \psi}(s, z(s), Y(s, z(s))) ds + u_\psi(0, z(0)) \right\}$$

where  $z$  solves the stochastic differential equation

$$\begin{cases} dz(s) = Y(s, z(s)) ds + \frac{1}{\sqrt{\beta}} dw(s) & s \geq t \\ z(t) = x \end{cases} \quad (SDE)_{t, Y, \delta_x}$$

and  $Y(t, z)$  varies among the vector fields continuous in  $t$  and Lipschitz in  $z$ . We have denoted by  $E_w$  the expectation with respect to the Wiener measure. From [9], we get that the minimal  $Y_\psi$  is given by

$$Y_\psi(t, x) = c - \partial_x u_\psi(t, x).$$

By (1.22), there is  $C_{12} > 0$  such that, for any  $\psi \in Den$ ,

$$\|Y_\psi\|_{C(\mathbf{T}, C^2(\mathbf{T}^p))} + \|Y_\psi\|_{C^1(\mathbf{T}, C(\mathbf{T}^p))} \leq C_{12}. \quad (1.24)$$

**Definition.** We group in a set  $Vect$  all the vector fields  $Y: \mathbf{T} \times \mathbf{T}^p \rightarrow \mathbf{R}^p$  which satisfy (1.24). The distance on  $Vect$  is given by the norm of (1.24).

We would like to consider the law of the stochastic differential equation above when the initial condition is distributed according to a measure  $\mu$ . One way to do this is to call  $\rho_{x_0}$  the solution of  $(FP)_{t, Y, \delta_{x_0}}$  and to set

$$\rho(s, x_0) = \int_{\mathbf{T}^p} \rho_{x_0}(s, x) d\mu(x_0).$$

Another one, which yields the same law, is to suppose that the Brownian motion is on a probability space  $\Omega$  on which there is a random variable  $M$  independent on  $w(s)$  for  $s \geq t$  and with law  $\mu$ ; we consider the solution  $z$  of the stochastic differential equation above with initial condition  $M$  and we say that  $z$  solves  $(SDE)_{t, Y, \mu}$ .

Let  $Y \in Vect$ ; by [13], there is  $\mu \in C(\mathbf{T}, \mathcal{M}_1(\mathbf{T}^p))$  which is invariant by the stochastic differential equation; in other words, there is a measure  $\mu_0$  such that, if  $\mu_t$  is the measure induced by a solution  $z$  of  $(SDE)_{0, Y, \mu_0}$  for  $t \geq 0$ , then  $\mu_0 = \mu_1$ . Equivalently, we are saying that there is a weak solution  $\mu$  of  $(FP)_{Y, per}$ . We sketch a proof of this fact: the map which brings the measure  $\mu_0$  into  $\mu_1$ , the solution of the Fokker-Planck equation at time 1, has a fixed point by the Schauder theorem.

We shall use the following classical uniqueness result ([7], proposition 1, [1], theorem 4.1, [17], theorem 5.34) to prove that  $\mu$  has a smooth density  $\rho_Y$ .

**Lemma 1.6.** *Let  $Y \in Vect$ . For  $i = 1, 2$ , let the map  $\nu^i: [0, +\infty) \rightarrow \mathcal{M}_1(\mathbf{T}^p)$  be continuous and let it be a weak solution of the Fokker-Planck equation, i. e.*

$$\int_{\mathbf{T}^p} \phi(0, x) d\nu_0^i(x) + \int_0^{+\infty} dt \int_{\mathbf{T}^p} \left[ \partial_t \phi + \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_x \phi \rangle \right] d\nu_t^i = 0 \quad (1.25)$$

for all  $\phi \in C_c^1([0, +\infty) \times \mathbf{T}^p) \cap C([0, +\infty), C^2(\mathbf{T}^p))$ . Let  $\nu_0^1 = \nu_0^2$ . Then,  $\nu_t^1 = \nu_t^2$  for all  $t \geq 0$ .

**Proof.** We begin to note that  $\mu_t := \nu_t^2 - \nu_t^1$  satisfies

$$\int_0^{+\infty} dt \int_{\mathbf{T}^p} \left[ \partial_t \phi + \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_x \phi \rangle \right] d\mu_s = 0 \quad \forall \phi \in C_c^1([0, +\infty) \times \mathbf{T}^p) \cap C([0, +\infty), C^2(\mathbf{T}^p)). \quad (1.26)$$

We have to prove that  $\mu_t = 0$  for all  $t \geq 0$ .

We define the operator  $A_Y$  as

$$A_Y \phi = \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_x \phi \rangle.$$

Let  $\gamma \in C_c^1([0, +\infty) \times \mathbf{T}^p)$  and let  $t$  be so large that  $\text{supp } \gamma \subset \subset [0, t) \times \mathbf{T}^p$ . The heat equation with time reversed and final condition in  $t$

$$\begin{cases} \partial_s \phi + A_Y \phi = \gamma & s < t \\ \phi(t, x) = 0 & \forall x \in \mathbf{T}^p \end{cases}$$

has a unique solution  $\phi$ . We set

$$\psi(s, x) = \begin{cases} \phi(s, x) & s \leq t \\ 0 & s > t \end{cases}$$

and we see that  $\psi \in C_c^1([0, +\infty) \times \mathbf{T}^p) \cap C([0, +\infty), C^2(\mathbf{T}^p))$ . Indeed,  $\psi$  is  $C^1$  in  $t$  and  $C^2$  in  $x$  on  $s < t$  by theorem 9 of chapter 1 of [10]; it is obviously  $C^2$  on  $s > t$ ; it is  $C^2$  also in a neighbourhood of  $s = t$ , because, by the uniqueness of the Cauchy problem for the equation  $\partial_s \phi + A_Y \phi = \gamma$ , and the fact that  $\gamma(s, x) = 0$  for  $s \in [t - \epsilon, t]$  we have that  $\phi(s, x) = 0$  for  $s \geq t - \epsilon$ . We use  $\psi$  as a test function in (1.26), getting the second equality below.

$$0 = \int_0^{+\infty} ds \int_{\mathbf{T}^p} [\partial_s \psi + A_Y \psi] d\mu_s(x) = \int_0^{+\infty} ds \int_{\mathbf{T}^p} \gamma(s, x) d\mu_s(x).$$

Since the formula above holds for all  $\gamma \in C_c^1([0, +\infty) \times \mathbf{T}^p)$ , we get the thesis.

\\

**Lemma 1.7.** *Let  $Y \in \text{Vect}$ . By [13], there is  $\mu \in C(\mathbf{T}, \mathcal{M}_1(\mathbf{T}^p))$  which solves  $(FP)_{Y, \text{per}}$  in the weak sense. Then, the following holds.*

- 1) *The measure  $\mu$  has density  $\rho_Y \in \text{Den}$ .*
- 2) *The measure  $\mu$  is unique.*
- 3) *There is  $C_{13} > 0$ , independent on  $Y \in \text{Vect}$ , such that*

$$\|\rho_Y\|_{C^1(\mathbf{T} \times \mathbf{T}^p)} + \|\rho_Y\|_{C(\mathbf{T}, C^2(\mathbf{T}^p))} \leq C_{13}.$$

- 4) *If  $Y_n \in \text{Vect}$  for all  $n$ , if  $Y \in \text{Vect}$  and  $Y_n \rightarrow Y$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ , then  $\rho_{Y_n} \rightarrow \rho_Y$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ .*

**Proof.** Classical results about PDE's (see lemma 2.3 below for more details) imply that there is a density  $\rho_{x_0}$ , smooth on  $(0, +\infty) \times \mathbf{T}^p$ , which solves

$$\begin{cases} \frac{1}{2\beta} \Delta \rho_{x_0} - \text{div}[\rho_{x_0} \cdot Y] - \partial_t \rho_{x_0} = 0 \\ \rho_{x_0}(t, \cdot) \mathcal{L}^p \rightarrow \delta_{x_0} \quad \text{as } t \rightarrow 0 \end{cases} \quad (FP)_{0, Y, \delta_{x_0}}$$

where  $\mathcal{L}^p$  denotes the Lebesgue measure on  $\mathbf{T}^p$ . It is standard that, for  $t > 0$ ,  $\rho_{x_0}(t, \cdot)$  satisfies properties d2) and d3) of the introduction, and that

$$\|\rho_{x_0}\|_{C^1([1, 2] \times \mathbf{T}^p)} + \|\rho_{x_0}\|_{C([1, 2], C^2(\mathbf{T}^p))} \leq C_{13} \quad (1.27)$$



for a constant  $C_{13} > 0$  which depends only on the  $C^1$  norm of  $Y$ ; as a consequence,  $C_{13}$  is the same for all  $Y \in Vect$  and  $x_0 \in \mathbf{T}^p$  (again, we refer the reader to lemma 2.3 below).

For  $t > 0$ , we define

$$\rho_Y(t, z) = \int_{\mathbf{T}^p} \rho_{x_0}(t, z) d\mu_0(x_0). \quad (1.28)$$

By lemma 1.6,  $\rho_Y(t, \cdot)$  is the density of  $\mu_t$ ; since  $\mu_t$  is periodic, we get that  $\rho_Y(t+1, \cdot) = \rho_Y(t, \cdot)$ . Point 3) follows from this, (1.27), (1.28) and the fact that norms are convex. One consequence of point 3) is that  $\rho_Y$  also satisfies hypothesis d1) of the introduction; since we saw above that it satisfies d2) and d3), we get that  $\rho_Y \in Den$ . Again from point 3), we get that  $\rho_Y$  is a classical solution of  $(FP)_{Y,per}$ ; since by [13] there is only one of them, we get point 2).

We prove point 4). Let  $Y_n \rightarrow Y$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ , and let  $\rho_{Y_n}$  and  $\rho_Y$  solve  $(FP)_{Y_n,per}$  and  $(FP)_{Y,per}$  respectively. We have just proved that  $\rho_{Y_n}$  satisfies point 3) of the thesis; thus, we can apply Ascoli-Arzelà and get that, up to subsequences,  $\rho_{Y_n} \rightarrow \rho$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ . Taking limits in (1.25) we see that  $\rho$  is a weak, periodic solution of  $(FP)_{Y,per}$ ; by the uniqueness of point 2), we get that  $\rho = \rho_Y$ . Thus, any subsequence of  $\rho_{Y_n}$  has a sub-subsequence converging to  $\rho_Y$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ ; by a well-known principle, this implies that  $\rho_{Y_n} \rightarrow \rho_Y$  in  $C(\mathbf{T} \times \mathbf{T}^p)$ .

\\

**Definition.** Let  $C_{13}$  be as in lemma 1.7. We group in a set  $\rho \in Den^{reg}$  the elements of  $Den$  which belong to  $Lip(\mathbf{T} \times \mathbf{T}^p)$  and such that  $\|\rho\|_{Lip(\mathbf{T} \times \mathbf{T}^p)} \leq C_{13}$ . By point 3) of lemma 1.7, if  $Y \in Vect$ , then  $\rho_Y \in Den^{reg}$ .

**Lemma 1.8.** *There is a continuous map  $\Phi: Den \rightarrow Den$  whose fixed points  $\rho_\beta$  induce solutions  $(u_{\rho_\beta}, \rho_\beta, \bar{H}_{\rho_\beta}(c))$  of  $(HJ)_{\rho_\beta,per} - (FP)_{c-\partial_x u_{\rho_\beta},per}$ . Moreover,  $\Phi(Den) \subset Den^{reg}$ .*

**Proof.** We define the map  $\Phi$  by composition. By lemma 1.5 and formula (1.24), we know that there is a map

$$\Phi_1: Den \rightarrow Vect \times \mathbf{R}, \quad \Phi_1: \psi \rightarrow (c - \partial_x u_\psi, \bar{H}_\psi(c)).$$

This map is continuous by point 3) of lemma 1.5.

Let  $\rho_Y$  be as in point 1) of lemma 1.7; by point 4) of this lemma, the map

$$\Phi_2: Vect \times \mathbf{R} \rightarrow Den, \quad \Phi_2: (Y, \lambda) \rightarrow \rho_Y$$

is continuous; by point 3), it has image in  $Den^{reg}$ . Thus, the map  $\Phi := \Phi_2 \circ \Phi_1$  is continuous from  $Den$  to  $Den$ , and has image in  $Den^{reg}$ , as we wanted.

Let now  $\rho_\beta$  be a fixed point of  $\Phi$ ; we recall that  $\Phi_1(\rho_\beta) = (c - \partial_x u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$ , with  $(u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$  which satisfies  $(HJ)_{\rho_\beta,per}$  and (1.21). Moreover,

$$\rho_\beta = \Phi_2 \circ \Phi_1(\rho_\beta) = \Phi_2(c - \partial_x u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$$

solves, by the definition of  $\Phi_2$ ,  $(FP)_{c-\partial_x u_{\rho_\beta}, per}$ ; in other words,  $(\rho_\beta, u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$  solves  $(HJ)_{\rho_\beta, per} - (FP)_{c-\partial_x u_{\rho_\beta}, per}$ , and we are done.

\\

**Proof of theorem 1.** We begin to show that there are couples  $(u_\beta, \rho_\beta)$  which satisfy  $(HJ)_{\rho_\beta, per} - (FP)_{c-\partial_x u_\beta, per}$ . By lemma 1.8, this follows if we show that  $\Phi$  has fixed points. But this is true by Schauder's fixed point theorem: indeed, by lemma 1.8,  $\Phi$  is a continuous map from  $Den$  to itself which preserves the compact, convex set  $Den^{reg}$ .

Let us now call  $\mathbf{S}$  the set of the triples  $(u, \rho, H)$  such that  $\rho \in Den$  is a weak solution of  $(FP)_{c-\partial_x u, per}$  and  $(u, H)$  is a classical solution of  $(HJ)_{\rho, per}$ . Let  $(u^n, \rho^n, H^n) \in \mathbf{S}$  be such that

$$\int_{\mathbf{T} \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho^n}(t, x, \partial_x u^n) \rho^n(t, x) dt dx \rightarrow \inf_{(u, \rho, H) \in \mathbf{S}} \int_{\mathbf{T} \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho}(t, x, \partial_x u) \rho(t, x) dt dx. \quad (1.29)$$

By lemma 1.1,  $\mathcal{L}_{c, \frac{1}{2}\rho}$  is bounded from below independently on  $\rho$ ; as a consequence, the inf in the right hand side of (1.29) is finite. Note that, if  $\rho^n \in Den$ , lemma 1.5 implies that  $c - \partial_x u^n \in Vect$ ; since  $\rho^n$  is a fixed point, we get by lemma 1.7 that  $\rho^n \in Den_m^{reg}$ ; since  $Den^{reg}$  is compact in  $Den$ , we can suppose that, up to subsequences,

$$\rho^n \rightarrow \bar{\rho} \quad \text{in } Den.$$

By point 3) of lemma 1.5, this implies that

$$(u^n, H^n) \rightarrow (\bar{u}, \bar{H}) \quad \text{in } [C(\mathbf{T}, C^3(\mathbf{T}^p)) \cap C^1(\mathbf{T}, C^1(\mathbf{T}^p))] \times \mathbf{R},$$

with  $(\bar{u}, \bar{H})$  solving  $(HJ)_{\bar{\rho}, per}$ . This and point 4) of lemma 1.7 yield that  $\bar{\rho} = \rho_{c-\partial_x \bar{u}}$  solves  $(FP)_{c-\partial_x \bar{u}, per}$  and satisfies the estimate of point 3) of that lemma. In other words,  $(\bar{u}, \bar{\rho}, \bar{H}) \in \mathbf{S}$ ; now (1.29) and the last three formulas easily imply that

$$\int_{\mathbf{T} \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(t, x, \partial_x \bar{u}) \bar{\rho}(t, x) dt dx = \inf_{(u, \rho, H) \in \mathbf{S}} \int_{\mathbf{T} \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho}(t, x, \partial_x u) \rho(t, x) dt dx$$

yielding the thesis.

\\

## §2

### The evolution equation

In this section, we shall prove theorems 2 and 3. We begin with some notation.

We recall that the map

$$: (\mu, \nu) \rightarrow d_1(\mu, \nu)$$

is convex, i. e.

$$d_1((1-\lambda)\nu_1 + \lambda\mu_1, (1-\lambda)\nu_2 + \lambda\mu_2) \leq (1-\lambda)d_1(\nu_1, \mu_1) + \lambda d_1(\nu_2, \mu_2).$$

Indeed, the dual formulation

$$d_1(\mu, \nu) = \sup \left\{ \int_{\mathbf{T}^p} f d(\mu - \nu) : f \in Lip^1(\mathbf{T}^p) \right\}$$

implies that  $d_1$  is the supremum of a family of linear functions. Since the functions  $f$  in the dual formulation belong to  $Lip_1(\mathbf{T}^p)$  and  $\mathbf{T}^p$  has diameter  $\sqrt{p}$ , we can as well suppose that  $\|f\|_\infty \leq \frac{1}{2}\sqrt{p}$ ; as a consequence,

$$d_1(\mu, \nu) \leq \sqrt{p} \|\mu - \nu\|_{tot}, \quad (2.1)$$

where  $\|\cdot\|_{tot}$  denotes total variation.

**Definition.** We are going to denote by the norm symbol the distance on  $C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$ , which is no norm at all: if  $R_1, R_2 \in C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$ , then we set

$$\|R_1 - R_2\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} = \sup_{t \in [-m, 0]} d_1(R_1(t), R_2(t)).$$

Though this is no norm, it is convex thanks to the convexity of  $d_1$ :

$$\begin{aligned} & \|(1 - \lambda)R_1 + \lambda R_2 - (1 - \lambda)\tilde{R}_1 - \lambda\tilde{R}_2\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} \leq \\ & (1 - \lambda)\|R_1 - \tilde{R}_1\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} + \lambda\|R_2 - \tilde{R}_2\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))}. \end{aligned} \quad (2.2)$$

**Definition.** For  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and  $m \in \mathbf{N}$ , we group in a set  $Den_m(\mu)$  all the maps  $R \in C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$  such that  $R(-m) = \mu$ . This space inherits the distance of  $C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$ .

**Lemma 2.1.** Let  $f \in C^3(\mathbf{T}^p)$  and let  $H^Z(t, q, p) = \frac{1}{2}|p|^2 + Z(t, q)$ , with  $Z \in C([-m, 0], C^3(\mathbf{T}^p))$ .

1) Then, there is a unique solution  $u^Z$  of

$$\begin{cases} \frac{1}{2\beta} \Delta u^Z + \partial_t u^Z - H^Z(t, x, c - \partial_x u^Z) = 0, & t \in [-m, 0] \\ u^Z(0, x) = f \quad \forall x \in \mathbf{T}^p. \end{cases} \quad (HJ)^Z$$

2) There is  $C_{13} > 0$ , only depending on  $\|f\|_{C^3(\mathbf{T}^p)}$ ,  $\|Z\|_{C([-m, 0], C^3(\mathbf{T}^p))}$  and  $m$ , such that

$$\|u^Z\|_{C^1([-m, 0], C^1(\mathbf{T}^p))} + \|u^Z\|_{C([-m, 0], C^3(\mathbf{T}^p))} \leq C_{13}.$$

3) The map

$$: Z \rightarrow u^Z$$

is continuous from  $C([-m, 0], C^3(\mathbf{T}^p))$  to  $C([-m, 0], C^3(\mathbf{T}^p)) \cap C^1([-m, 0], C^1(\mathbf{T}^p))$ .

**Proof.** We know that the twisted Schroedinger equation with potential  $Z$  and final condition  $e^{-\beta f} \in C^3(\mathbf{T}^p)$  has a unique solution  $v^Z$ , which can be represented by the Feynman-Kac formula (1.20) with  $e^{-\beta f}$

in stead of  $v_\psi$  and  $Z$  in stead of  $P_\psi$ . Since  $e^{-\beta f} > 0$ , we get that  $v^Z > 0$  too. We saw in section 1 that  $u^Z = -\frac{1}{\beta} \log v^Z$  solves  $(HJ)^Z$ , and point 1) follows.

Points 2) and 3) follow as in section 1 if we prove that

$$\|\partial_x v^Z\|_{C^1([-m,0],C^1(\mathbf{T}^p))} + \|v^Z\|_{C([-m,0],C^3(\mathbf{T}^p))} \leq C_{14}$$

and that the map  $Z \rightarrow (v^Z, \partial_x v^Z)$  is continuous. Since  $e^{-\beta f}$ , the final condition of the Schroedinger equation, is of class  $C^3$ , this is a standard result; for instance, differentiation under the integral sign in (1.20) gives the estimate on  $\|v^Z\|_{C([-m,0],C^3(\mathbf{T}^p))}$ ; from this and the fact that  $v^Z$  solves the Schroedinger equation, we get the estimate on  $\|v^Z\|_{C^1([-m,0],C^1(\mathbf{T}^p))}$ .

\\

Recalling lemma 1.1, we get this immediate consequence.

**Corollary 2.2.** 1) Let  $f \in C^3(\mathbf{T}^p)$ , let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and let  $R \in Den^m(\mu)$ . Then, there is a unique solution  $u_R$  of

$$\begin{cases} \frac{1}{2\beta} \Delta u_R + \partial_t u_R - H_R(t, x, c - \partial_x u_R) = 0, & t \in [-m, 0] \\ u_R(0, x) = f & \forall x \in \mathbf{T}^p. \end{cases} \quad (HJ)_{R,f}$$

2) There is  $C_{14} = C_{14}(m) > 0$ , independent of  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and on  $R \in Den_m(\mu)$ , such that

$$\|u_R\|_{C^1([-m,0],C^1(\mathbf{T}^p))} + \|u_R\|_{C([-m,0],C^3(\mathbf{T}^p))} \leq C_{14}.$$

3) The map

$$: R \rightarrow u_R$$

is continuous from  $Den_m(\mu)$  to  $C([-m,0],C^3(\mathbf{T}^p)) \cap C^1([-m,0],C^1(\mathbf{T}^p))$ .

**Definition.** By point 2) of corollary 2.2, there is  $C_{15} > 0$  such that, setting  $Y = c - \partial_x u_R$ , we have

$$\|Y\|_{C^1([-m,0],C(\mathbf{T}^p))} + \|Y\|_{C([-m,0],C^2(\mathbf{T}^p))} \leq C_{15}$$

with  $C_{15}$  independent on  $R \in C([-m,0],\mathcal{M}_1(\mathbf{T}^p))$ . We group in a set  $Vect_m$  all the vector fields  $Y$  on  $[-m,0] \times \mathbf{T}^p$  which satisfy the estimate above. The distance on  $Vect_m$  is the one induced by the norm above.

**Lemma 2.3.** Let  $Y \in Vect_m$ , and let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ . Then, the following holds.

1) There is a unique  $R_Y \in C([-m,0],\mathcal{M}_1(\mathbf{T}^p))$  which solves  $(FP)_{-m,Y,\mu}$  in the weak sense.

2) For  $t \in (-m,0]$ ,  $R_Y(t)$  has density  $\rho_Y$ . There are  $C_{16}, C_{17}: (-m,0] \rightarrow [0,+\infty)$ , independent on  $Y \in Vect_m$  and on  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ , such that

a)  $C_{17}(T) \rightarrow 0$  as  $T \rightarrow -m$ ,  $C_{16}$  and  $C_{17}$  are bounded on  $(-m+\epsilon,0]$  for all  $\epsilon \in (0,m)$  and

b) For  $T \in (-m, 0]$ , we have

$$\begin{cases} \|\rho_Y\|_{C^1((T,0) \times \mathbf{T}^p)} + \|\rho_Y\|_{C((T,0), C^2(\mathbf{T}^p))} \leq C_{16}(T) \\ d_1(R_Y(T), \mu) \leq C_{17}(T) \end{cases} \quad (2.3)$$

where  $d_1$  denotes the 1-Wasserstein distance.

**Proof.** The uniqueness of point 1) comes from lemma 1.6; for the existence, we begin to recall from PDE theory (see for instance chapter 1 of [10]) that, for  $x_0 \in \mathbf{T}^p$ ,  $(FP)_{-m, Y, \delta_{x_0}}$  has a solution  $R_{x_0}$  with density  $\rho_{x_0}$ . Always from [10], the function  $\rho_{x_0}$  satisfies the first formula of (2.3) for a constant  $C_{16}(T)$  which depends neither on  $x_0 \in \mathbf{T}^p$  nor on the particular element  $Y \in Vect_m$ . Moreover, as  $T \rightarrow -m$ , we get from [10] that, if  $g \in C(\mathbf{T}^p)$ , then

$$\int_{\mathbf{T}^p} g(x) dR_{x_0}(T) \rightarrow g(x_0)$$

uniformly in  $x_0 \in \mathbf{T}^p$ ; since  $d_1$  induces the weak\* topology and  $\mathbf{T}^p$  is compact, we have that  $d_1(R_{x_0}(T), \delta_{x_0}) \leq C_{17}(T)$ , for a constant  $C_{17}(T)$  which depends neither on  $x_0 \in \mathbf{T}^p$  nor on  $Y \in Vect_m$ , and such that  $C_{17}(T) \rightarrow 0$  as  $T \rightarrow -m$ . In other words,  $\rho_{x_0}$  satisfies (2.3) for two uniform constants  $C_{16}(T)$ ,  $C_{17}(T)$ , depending neither on  $x_0$  nor on  $Y \in Vect_m$ .

Now we set

$$\rho_Y(t, x) = \int_{\mathbf{T}^p} \rho_{x_0}(t, x) d\mu(x_0). \quad (2.4)$$

Clearly,  $\rho_Y$  is a solution of  $(FP)_{-m, Y, \mu}$ , and this ends the proof of point 1).

We have seen that  $\rho_{x_0}$  satisfies the first formula of (2.3); since norms are convex, (2.4) implies that  $\rho_Y$  too satisfies this formula. Now  $\rho_{x_0}$  satisfies  $d_1(R_{x_0}(T), \delta_{x_0}) \leq C_{17}(T)$ , and the map

$$: (\mu, \nu) \rightarrow d_1(\mu, \nu)$$

is convex; it follows again by (2.4) that  $\rho_Y$  too satisfies the second formula of (2.3).

\\

**Definition.** We define  $Den_m^{reg}(\mu)$  as the subset of the elements  $R \in Den_m(\mu)$  which, for  $t \in (-m, 0]$ , have a density  $\rho$  with respect to the Lebesgue measure. Moreover, we ask that  $R$  and  $\rho$  satisfy

$$\begin{cases} \|\rho\|_{Lip([T,0] \times \mathbf{T}^p)} \leq C_{16}(T), \quad \forall T \in (-m, 0] \\ d_1(R(T), \mu) \leq C_{17}(T), \quad \forall T \in (-m, 0] \end{cases} \quad (2.5)$$

where  $C_{16}(T)$  and  $C_{17}(T)$  are the same two constants of (2.3). By lemma 2.3, if  $Y \in Vect$ ,  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and  $R_Y$  solves  $(FP)_{-m, Y, \mu}$  in the weak sense, then  $R_Y \in Den_m^{reg}(\mu)$ .

**Lemma 2.4.**  $Den_m^{reg}(\mu)$  is compact in  $Den_m(\mu)$  for the  $C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$  topology.

**Proof.** Let  $R_n \in Den_m^{reg}(\mu)$  have density  $\rho_n$  for  $n \in \mathbf{N}$ . We must show that it has a subsequence converging in  $Den_m(\mu)$ .

Since  $\rho_n$  satisfies the first formula of (2.5), Ascoli-Arzelà implies that, up to subsequences,  $\rho_n \rightarrow \rho$  in  $C_{loc}^0((-m, 0] \times \mathbf{T}^p)$ ; clearly,  $\rho$  satisfies the first formula of (2.5). Denoting by  $\mathcal{L}^p$  the Lebesgue measure on  $\mathbf{T}^p$ , we set  $R(t) = \rho(t)\mathcal{L}^p$  and we see that, for any fixed  $T \in (-m, 0]$ ,

$$d_1(R_n(T), R(T)) \leq \sqrt{p} \|R_n(T) - R(T)\|_{tot} = \sqrt{p} \|\rho_n(T) - \rho(T)\|_{L^1(\mathbf{T}^p)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

where the first inequality comes from (2.1) and the limit from the fact that  $\rho_n \rightarrow \rho$  in  $C_{loc}^0((-m, 0] \times \mathbf{T}^p)$ . Since  $R_n$  satisfies the second formula of (2.5), we have that

$$d_1(R_n(T), \mu) \leq C_{17}(T), \quad \forall T \in (-m, 0], \quad \forall n \geq 1.$$

The last two formulas imply that  $R$  satisfies the second formula of (2.5).

It remains to prove that  $R_n \rightarrow R$  in  $C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$ ; it suffices to note that, for  $\delta \in (0, m)$ ,

$$\begin{aligned} \sup_{t \in [-m, 0]} d_1(R_n(t), R(t)) &\leq \\ \sup_{t \in [-m, -m+\delta]} [d_1(R_n(t), \mu) + d_1(\mu, R(t))] &+ \sup_{t \in [-m+\delta, 0]} d_1(R_n(t), R(t)) \leq \\ 2C_{17}(-m+\delta) + \sqrt{p} \sup_{t \in [-m+\delta, 0]} \|\rho_n(t) - \rho(t)\|_{L^1(\mathbf{T}^p)} \end{aligned}$$

where the last inequality comes from the second formula of (2.5) and from (2.1). Since  $C_{17}(T) \rightarrow 0$  as  $T \searrow -m$ , we can fix  $\delta > 0$  so that the first term on the right is smaller than  $\epsilon$ ; having thus fixed  $\delta$ , we take  $n$  so large that, by convergence in  $C_{loc}^0((-m, 0] \times \mathbf{T}^p)$ , the second term on the right is smaller than  $\epsilon$ , and we are done.

\\

We only sketch the proof of the next lemma, since it is identical to point 4) of lemma 1.7.

**Lemma 2.5.** *Given  $\epsilon > 0$ , we can find  $\delta > 0$  with the following property. Let  $Y, \bar{Y} \in Vect_m$  and let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ ; let  $R_{\bar{Y}}$  and  $R_Y$  satisfy  $(FP)_{-m, \bar{Y}, \mu}$  and  $(FP)_{-m, Y, \mu}$  respectively. Let  $\|\bar{Y} - Y\|_{C([-m, 0] \times \mathbf{T}^p)} \leq \delta$ . Then,  $\|R_{\bar{Y}} - R_Y\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} \leq \epsilon$ .*

**Proof.** Let  $\{Y_n\}_{n \geq 1}$ ,  $\{\bar{Y}_n\}_{n \geq 1}$  be two sequences in  $Vect_m$  and let  $\{\mu_n\}_{n \geq 1} \subset \mathcal{M}_1(\mathbf{T}^p)$ . We suppose that  $\|\bar{Y}_n - Y_n\|_{C([-m, 0] \times \mathbf{T}^p)} \rightarrow 0$ ; we let  $R_{Y_n}$  solve  $(FP)_{-m, Y_n, \mu_n}$  and  $R_{\bar{Y}_n}$  solve  $(FP)_{-m, \bar{Y}_n, \mu_n}$ ; we have to prove that

$$\|R_{\bar{Y}_n} - R_{Y_n}\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} \rightarrow 0.$$

Let us suppose by contradiction that this does not hold; then there is  $\epsilon > 0$  and a subsequence (which we denote by the same index) such that

$$\|R_{\bar{Y}_n} - R_{Y_n}\|_{C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))} > \epsilon \quad \forall n.$$

Since  $\bar{Y}_n, Y_n \in Vect_m$  and  $\|\bar{Y}_n - Y_n\|_{C([-m, 0] \times \mathbf{T}^p)} \rightarrow 0$ , by Ascoli-Arzelà up to taking subsequences we can suppose that  $\bar{Y}_n, Y_n \rightarrow Y$  in  $C([-m, 0] \times \mathbf{T}^p)$ ; we can also suppose that  $\mu_n \rightarrow \mu$ . To reach a contradiction

with the formula above, it suffices to show that  $R_{Y_n}$  and  $R_{\bar{Y}_n}$  both converge to  $R_Y$ ; since the proof for  $R_{\bar{Y}_n}$  is analogous, we prove convergence for  $R_{Y_n}$ .

We note that  $\{R_{Y_n}\}$  is contained in  $Den_m^{reg}(\mu_n)$  by lemma 2.3; thus, by lemma 2.4, it has a subsequence converging to a limit  $R$ . Since  $R_{Y_n}$  is a weak solution of  $(FP)_{-m, Y_n, \mu_n}$ , we easily get that  $R$  is a weak solution of  $(FP)_{-m, Y, \mu}$ ; by lemma 1.6,  $R = R_Y$ . In other words, every subsequence of  $R_{Y_n}$  has a sub-subsequence converging to  $R_Y$ ; this implies that  $R_{Y_n}$  converges to  $R_Y$ , and we are done.

\\

**Proof of theorem 2.** For  $Q \in Den_m(\mu)$ , let  $u_Q$  be as in corollary 2.2; for  $Y \in Vect_m$ , let  $R_Y = \rho_Y \mathcal{L}^p$  be as in lemma 2.3. The two maps

$$: Den_m(\mu) \rightarrow Vect_m, \quad : Q \rightarrow c - \partial_x u_Q$$

and

$$: Vect_m \rightarrow Den_m(\mu), \quad : Y \rightarrow R_Y$$

are both continuous: the first one, by point 3) of corollary 2.2, the second one by lemma 2.5. Let us call  $\Phi$  their composition:

$$\Phi: Den_m(\mu) \rightarrow Den_m(\mu), \quad \Phi: Q \rightarrow R_{c - \partial_x u_Q}.$$

Being the composition of two continuous functions,  $\Phi$  is continuous; moreover, by point 2) of lemma 2.3, it has image in  $Den_m^{reg}(\mu)$ ; this latter set is clearly convex, and it is compact in  $Den_m(\mu)$  by lemma 2.4. Thus, we have that

$$\Phi: Den_m^{reg}(\mu) \rightarrow Den_m^{reg}(\mu).$$

We apply the Schauder fixed point theorem and we get that  $\Phi$  has a fixed point in  $Den_m^{reg}(\mu)$ . With the same argument as in the proof of theorem 1, we see that, if  $R$  is a fixed point of  $\Phi$ , then  $(u_R, R)$  solves  $(HJ)_{R,f} - (FP)_{-m, c - \partial_x u_R, \mu}$ . This yields existence.

We continue as in the proof of theorem 1. Let us call  $\mathbf{S}$  the set of the couples  $(u, R)$  where  $u$  is a classical solution of  $(HJ)_{R,f}$  and  $R \in Den_m(\mu)$  is a weak solution of  $(FP)_{-m, c - \partial_x u, \mu}$ .

Let us consider a sequence  $(u_n, R_n) \in \mathbf{S}$  such that, denoting by  $\rho_n$  the density of  $R_n$ ,

$$\int_{-m}^0 dt \int_{\mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}R_n}(t, x, c - \partial_x u_n) \rho_n dt dx \rightarrow \inf_{(u, R) \in \mathbf{S}} \int_{-m}^0 dt \int_{\mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}R}(t, x, c - \partial_x u) \rho dt dx.$$

Whatever is  $R_n \in Den_m(\mu)$ ,  $u_n$  satisfies the estimates of point 2) of corollary 2.2; in particular,  $c - \partial_x u_n \in Vect_m$ . Since  $R_n$  satisfies  $(FP)_{-m, c - \partial_x u_n, \mu}$  lemma 2.3 implies that  $R_n \in Den_m^{reg}(\mu)$ ; by lemma 2.4, up to subsequences we can suppose that  $R_n \rightarrow \bar{R}$ , with  $\bar{R} \in Den_m^{reg}(\mu)$ . By point 3) of corollary 2.2, we get that  $u_n \rightarrow \bar{u}$  in  $C^1([-m, 0], C^1(\mathbf{T}^p)) \cap C([-m, 0], C^3(\mathbf{T}^p))$ , and that  $\bar{u}$  solves  $(HJ)_{\bar{R}, f}$ . Thus,  $(\bar{u}, \bar{R}) \in \mathbf{S}$ ; now, the formula above easily implies that  $(\bar{u}, \bar{R})$  is minimal in  $\mathbf{S}$ .

\\

We turn to the proof of theorem 3; our route will pass through an approximation with a finite number of particles.

**Definitions.** Let us define the Lagrangian for one particle as

$$L_c: \mathbf{T} \times \mathbf{T}^p \times \mathbf{R}^p \rightarrow \mathbf{R}, \quad L_c(t, x, y) = \frac{1}{2}|y|^2 - \langle c, y \rangle - V(t, x).$$

The Lagrangian for  $n$  particles, each of mass  $\frac{1}{n}$ , is

$$L_c^n: \mathbf{T} \times (\mathbf{T}^p)^n \times (\mathbf{R}^p)^n \rightarrow \mathbf{R}$$

$$L_c^n(t, (x_1, \dots, x_n), (y_1, \dots, y_n)) = \frac{1}{n} \sum_{i=1}^n L_c(t, x_i, y_i) + \frac{1}{2n^2} \sum_{i,j=1}^n W(x_i - x_j).$$

Let  $U$  be as in the statement of theorem 2. For any given  $z = (z_1, \dots, z_n) \in (\mathbf{T}^p)^n$ , we define

$$U^n(-m, z) = \inf_{E_{w_1, \dots, w_n}} \left\{ \int_{-m}^0 L_c^n(s, X^n(-m, s, z), Y^n(s, X^n(-m, s, z))) ds + U(R^n(-m, 0)) \right\}. \quad (2.6)$$

The infimum above is over all vector fields  $Y^n(s, x) = (Y_1^n(s, x_1), Y_2^n(s, x_2), \dots, Y_n^n(s, x_n))$  continuous in  $s$  and Lipschitz in  $x$ ; each component of the function

$$X^n(-m, s, z) = (X_1^n(-m, s, z_1), X_2^n(-m, s, z_2), \dots, X_n^n(-m, s, z_n)) \in (\mathbf{R}^p)^n$$

solves the stochastic differential equation on  $\mathbf{R}^p$

$$\begin{cases} dX_i^n(-m, s, z_i) = Y_i^n(s, X_i^n(-m, s, z_i))dt + dw_i(s) & s \geq -m, \quad i \in (1, \dots, n) \\ X_i^n(-m, -m, x_i) = z_i. \end{cases} \quad (SDE)_{t, Y_i^n, \delta_{z_i}}$$

In the formula above, each  $w_i$  is a standard Brownian motion on  $\mathbf{R}^p$ ; the  $w_i$  are independent and  $E_{w_1, \dots, w_n}$  denotes the expectation with respect to the product of the Wiener measures. It remains to define  $R^n(-m, 0)$ ; to do this, we let  $\rho_i^n(-m, s, x)$  be the density on  $\mathbf{T}^p$  which solves  $(FP)_{-m, Y_i^n, \delta_{z_i}}$  and we set, for  $t \in [-m, 0]$ ,

$$\rho^n(-m, t, x) = \frac{1}{n} \sum_{i=1}^n \rho_i^n(-m, t, x), \quad R^n(-m, t) = \rho^n(-m, t, x) \mathcal{L}^p. \quad (2.7)$$

We note that we are not considering the most general vector field  $Y$  on  $(\mathbf{T}^p)^n$ . On the contrary, we assign to each particle  $x_i \in \mathbf{T}^p$  a control  $Y_i$  which depends only on  $x_i$ , and not on the positions of the other particles; these, however, interact with  $x_i$  via the potential  $W$ . We have chosen this particular problem because we want  $U_n(-m, z)$  to converge, as  $n \rightarrow +\infty$ , to  $\Lambda_c^m U$ ; we recall that, in the definition of  $\Lambda_c^m$ , there is a control  $Y$  which depends on the single particle in  $\mathbf{T}^p$ .

**Lemma 2.6.** *Let us suppose that  $U$  is as in the statement of theorem 1 and let  $U^n(-m, z)$  be defined as in (2.6). Then for any fixed  $n \in \mathbf{N}$ , the infimum in (2.6) is a minimum.*



**Proof.** Let  $\{Y^{n,k}\}_{k \geq 1}$  be a minimizing sequence. We are going to show that we can build another minimizing sequence, say  $\{\tilde{Y}^{n,k}\}_{k \geq 1}$ , which is Lipschitz in  $(t, x)$  uniformly in  $k$ . Once we know this, the lemma follows by Ascoli-Arzelà.

For the vector field  $Y^{n,k}$ , let us define  $\rho_i^{n,k}$  and  $\rho^{n,k}$  as in (2.7); we set

$$L_{c,Y^{n,k},i}^n: [-m, 0] \times \mathbf{T}^p \times \mathbf{R}^p \rightarrow \mathbf{R},$$

$$L_{c,Y^{n,k},i}^n(s, x, \dot{x}) = L_c(s, x, \dot{x}) - \frac{1}{n} \sum_{j \neq i} \int_{\mathbf{T}^p} W(x - y) \rho_j^{n,k}(-m, s, y) dy. \quad (2.8)$$

Note that, in contrast with  $L_c^n$ , a factor  $\frac{1}{2}$  in the interaction sum is missing. We know from lemma 1.1 that the potential in  $L_{c,Y^{n,k},i}^n$  satisfies a uniform  $C^3$  estimate. By [9], for  $(t, x) \in [-m, 0] \times \mathbf{T}^p$ , there is  $\tilde{Y}_i^{n,k}$  on which the minimum below is attained

$$u_i^{n,k}(t, x) := \min E_w \left\{ \int_t^0 L_{c,Y^{n,k},i}^n(s, X, Y) ds + f(X(t, 0, x)) \right\} \quad (2.9)$$

with  $X(t, s, x)$  which solves  $(SDE)_{t,Y,\delta_x}$ ; the minimum is taken over all the Lipschitz vector fields  $Y$ . Always by [9],  $\tilde{Y}_i^{n,k} = c - \partial_x u_i^{n,k}(t, x)$  and  $u_i^{n,k}$  solves the Hamilton-Jacobi equation for the Lagrangian  $L_{c,Y^{n,k},i}^n$  and final condition  $f$ . By lemma 2.1,

$$\|u_i^{n,k}\|_{C^1([-m,0], C^1(\mathbf{T}^p))} + \|u_i^{n,k}\|_{C([-m,0], C^3(\mathbf{T}^p))}$$

is bounded in terms of the  $C^3$  norm of the potential of  $L_{c,Y^{n,k},i}^n$ . By lemma 1.1, the latter depends neither on  $n$  nor on  $k$ ; thus,  $\tilde{Y}_i^{n,k}$  belongs to  $Vect_m$ ; in particular, it is Lipschitz uniformly in  $n$  and  $k$ .

In the following, whenever we have a drift, say  $Y_i^B$ , we shall denote by  $X_i^B(t, s, x_i)$  the solution of  $(SDE)_{t,Y_i^B,\delta_{x_i}}$ ; we shall set  $X^B = (X_1^B, \dots, X_n^B)$  and  $z = (z_1, \dots, z_n)$ .

We are going to isolate the first particle and show that the mean action decreases if we substitute  $Y_1^{n,k}$  with the smoother  $\tilde{Y}_1^{n,k}$  defined above. Since the interaction potential is even and satisfies  $W(0) = 0$ , we get the first equality below; since the Brownian motions  $(w_1, \dots, w_n)$  are independent, we get the second one.

$$\begin{aligned} E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, X^{n,k}(-m, s, z), Y^{n,k}(s, X^{n,k}(-m, s, z))) ds + U(R^n(-m, 0)) \right\} = \\ E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^0 L_c(s, X_j^{n,k}(-m, s, z_j), Y_j^{n,k}(s, X^{n,k}(-m, s, z_j))) ds + \right. \\ \left. \frac{1}{2n^2} \sum_{j, i \neq 1} \int_{-m}^0 W(X_i^{n,k}(-m, s, z_i) - X_j^{n,k}(-m, s, z_j)) ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m, 0, z_j)) \right\} + \\ E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \int_{-m}^0 [L_c(s, X_1^{n,k}(-m, s, z_1), Y_1^{n,k}(s, X_1^{n,k}(-m, s, z_1))) + \right. \\ \left. \frac{1}{n^2} \sum_{j \neq 1} \int_{-m}^0 W(X_1^{n,k}(-m, s, z_1) - X_j^{n,k}(-m, s, z_j))] ds + \frac{1}{n} f(X_1(-m, 0, z_1)) \right\} = \end{aligned}$$

$$a1) \quad E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^0 L_c(s, X_j^{n,k}(-m, s, z_j), Y_j^{n,k}(s, X_j^{n,k}(-m, s, z_j))) ds + \right.$$

$$a2) \quad \left. \frac{1}{2n^2} \sum_{j, i \neq 1} \int_{-m}^0 W(X_i^{n,k}(-m, s, z_i) - X_j^{n,k}(-m, s, z_j)) ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m, 0, z_j)) \right\} +$$

$$a3) \quad \frac{1}{n} E_{w_1} \left\{ \int_{-m}^0 L_{c, Y^{n,k}, 1}^n(s, X_1^{n,k}(-m, s, z_1), Y_1^{n,k}(s, X_1^{n,k}(-m, s, z_1))) ds + f(X_1(-m, 0, z_1)) \right\}.$$

If we consider  $(\tilde{Y}_1^{n,k}, \tilde{Y}_2^{n,k}, \dots, \tilde{Y}_n^{n,k})$  instead of  $(Y_1^{n,k}, Y_2^{n,k}, \dots, Y_n^{n,k})$ , we see that the terms *a1*) and *a2*) in the formula above remain the same, while, by our choice of  $\tilde{Y}_1^{n,k}$ , *a3*) gets smaller. After applying this procedure to each coordinate, we get a sequence  $\tilde{Y}^{n,k} = (\tilde{Y}_1^{n,k}, \dots, \tilde{Y}_n^{n,k})$  which satisfies the following two properties.

- $E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, \tilde{X}^{n,k}(-m, s, x), \tilde{Y}^{n,k}(s, \tilde{X}^{n,k}(-m, s, x))) ds + \frac{1}{n} \sum_{j=1}^n f(\tilde{X}^{n,k}(-m, s, x)) \right\} \leq$

$$E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, X^{n,k}(-m, s, x), Y^{n,k}(s, X^{n,k}(-m, s, x))) ds + \frac{1}{n} \sum_{j=1}^n f(X^{n,k}(-m, s, x)) \right\}.$$

- $\tilde{Y}^{n,k} \in (Vect_m)^n$ .

In particular,  $\tilde{Y}_i^{n,k} \in Vect_m$  for all  $i, n$  and  $k$ ; as a consequence, we can apply point 2) of lemma 2.3, getting that  $\{\rho_i^{n,k}\}_{n,k} \subset Den_m^{reg}(\delta_{z_i})$  for all  $i$ . By lemma 2.4, we find a subsequence, which we denote by the same index, such that

$$(\rho_1^{n,k} \mathcal{L}^p, \dots, \rho_n^{n,k} \mathcal{L}^p) \rightarrow (\rho_1^n \mathcal{L}^p, \dots, \rho_n^n \mathcal{L}^p) \quad \text{in} \quad Den_m(\delta_{z_1}) \times \dots \times Den_m(\delta_{z_n}).$$

Thus, for each  $i$ ,

$$\frac{1}{n} \sum_{j \neq i} \int_{\mathbf{T}^p} W(x - y) \rho_j^{n,k}(-m, s, y) dy \rightarrow \frac{1}{n} \sum_{j \neq i} \int_{\mathbf{T}^p} W(x - y) \rho_j^n(-m, s, y) dy$$

in  $C([-m, 0], C^3(\mathbf{T}^p))$ . By point 3) of lemma 2.1, this implies that

$$(\tilde{Y}_1^{n,k}, \dots, \tilde{Y}_n^{n,k}) \rightarrow (\tilde{Y}_1^n, \dots, \tilde{Y}_n^n) \quad \text{in} \quad C^1([-m, 0], C(\mathbf{T}^p)) \cap C([-m, 0], C^2(\mathbf{T}^p))$$

and that each  $\tilde{Y}_i^n$  is minimal for  $L_{c, \tilde{Y}, i}^n$ . By the last formula and lemma 2.5, we get that  $\rho_i^n$  solves  $(FP)_{-m, \tilde{Y}_i^n, \delta_{z_i}}$ . The last three formulas imply that

$$E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, \tilde{X}^n(-m, s, z), \tilde{Y}^n(s, \tilde{X}^n(-m, s, z))) ds + U(R^n(-m, 0, z)) \right\} =$$

$$\lim_{n \rightarrow +\infty} E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, \tilde{X}^{n,k}(-m, s, z), \tilde{Y}^{n,k}(s, \tilde{X}^{n,k}(-m, s, z))) ds + U(R^{n,k}(-m, 0, z)) \right\}.$$

Since  $\{(\tilde{Y}_1^{n,k}, \dots, \tilde{Y}_n^{n,k})\}_{k \geq 1}$  is a minimizing sequence, we get that  $(\tilde{Y}_1^n, \dots, \tilde{Y}_n^n)$  is minimal.

\\

From the proof of the last lemma, we extract the following corollary: it says that the minimum in (2.6) is a Nash equilibrium ([4]). Note one fact about the value function  $u_i^n$  in the corollary below: for simplicity, we let  $i = 1$ . Then the function  $u_1^n$  depends not only on  $(x_2, \dots, x_n)$ , but on  $x_1$  too: namely, if  $x_1$  moves, the drifts  $(Y_2, \dots, Y_n)$  will adjust, and the Lagrangian  $L_{c,Y,1}$  will change. If it hadn't been too clumsy, we could have written  $u_1^{n,(x_1, \dots, x_n)}$  and said that  $c - \partial_x u_1^{n,(x_1, \dots, x_n)}(x)$  is the best drift for particle  $x_1$ .

**Corollary 2.7.** *Let  $\bar{Y}^n(t, x) = (\bar{Y}_1^n(t, x_1), \dots, \bar{Y}_n^n(t, x_n))$  be minimal in (2.6) and let  $L_{c, \bar{Y}^n, i}^n$  be defined as in (2.8). Let*

$$u_i^n(t, x) = \min_Y E_w \left\{ \int_t^0 L_{c, Y, i}^n(s, X, Y) ds + f(X(t, 0, x)) \right\}$$

where  $X$  solves  $(SDE)_{t, Y, \delta_x}$  and the minimum is taken among all Lipschitz vector fields  $Y$  on  $[-m, 0] \times \mathbf{T}^p$ . Then, for each  $i$  we have that  $\bar{Y}_i^n(t, x_i) = c - \partial_x u_i^n(t, x_i)$ .

**Proof.** If for one  $i$  we had  $\bar{Y}_i^n \neq c - \partial_x u_i^n$ , then, isolating particle  $i$  as in the last lemma, we could see that the vector field

$$(\bar{Y}_1^n, \dots, \bar{Y}_{i-1}^n, c - \partial_x u_i^n, \bar{Y}_{i+1}^n, \dots, \bar{Y}_n^n)$$

has a lower Lagrangian action, contradicting the minimality of  $\bar{Y}^n$ .

\\

**Lemma 2.8.** *Let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and let us suppose that*

$$\frac{1}{n}(\delta_{z_1} + \dots + \delta_{z_n}) \rightarrow \mu \quad \text{in } \mathcal{M}_1(\mathbf{T}^p). \quad (2.10)$$

Let  $Y^n = (Y_1^n, \dots, Y_n^n)$  be a drift minimal in (2.6); by corollary 2.7,  $Y_i^n = c - \partial_x u_i^n$  for the value function  $u_i^n$  defined in (2.9). Let  $\rho^n$  be defined as in (2.7).

Then, there is  $(u, \rho)$  which satisfies  $(HJ)_{\rho, f} - (FP)_{-m, c - \partial_x u, \mu}$ , and a subsequence  $\{n_k\}$  such that

$$\left\{ \begin{array}{l} \rho^{n_k} \mathcal{L}^p \rightarrow \rho \mathcal{L}^p \quad \text{in } C([-m, 0], \mathcal{M}_1(\mathbf{T}^p)) \\ \sup_{0 \leq i \leq n_k} \|u_i^{n_k} - \partial_x u\|_{C^1([-m, 0], C^1(\mathbf{T}^p))} + \|u_i^{n_k} - u\|_{C([-m, 0], C^3(\mathbf{T}^p))} \rightarrow 0 \\ \sup_{0 \leq i \leq n_k} \|Y_i^{n_k} - (c - \partial_x u)\|_{C^1([-m, 0], C(\mathbf{T}^p))} + \|Y_i^{n_k} - (c - \partial_x u)\|_{C([-m, 0], C^2(\mathbf{T}^p))} \rightarrow 0. \end{array} \right. \quad (2.11)$$

Moreover, the function  $U^{n_k}(-m, z_1, \dots, z_{n_k})$  defined in (2.6) converges to the function  $U(-m, \mu)$  defined by

$$U(-m, \mu) := \int_{[-m, 0] \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho}(t, x, c - \partial_x u) \rho(t, x) dx dt + U(\rho). \quad (2.12)$$

**Proof.** Since  $Y_i^n = c - \partial_x u_i^n$ , the third formula of (2.11) follows from the second one; we prove the first two ones.

**Step 1.** We prove the convergence of the densities.

For  $i, j \in (1, \dots, n)$ , we consider the densities

$$\begin{aligned}\hat{\rho}_i^n(-m, s, x) &= \frac{1}{n-1} \sum_{l \neq i} \rho_l^n(-m, s, x), & \hat{\rho}_j^n(-m, s, x) &= \frac{1}{n-1} \sum_{l \neq j} \rho_l^n(-m, s, x) \\ \rho^n(-m, s, x) &= \frac{1}{n} \sum_{l=1}^n \rho_l^n(-m, s, x)\end{aligned}\tag{2.13}$$

where  $\rho_l^n$  is the same as in formula (2.7). Let  $R_i^n = \rho_i^n \mathcal{L}^p$ ,  $\hat{R}_i^n = \hat{\rho}_i^n \mathcal{L}^p$  and  $R^n = \rho^n \mathcal{L}^p$ . Formula (2.1) implies the first inequality below, while the second one follows from the fact that  $\rho_j^n$  and  $\rho_i^n$  are probability densities.

$$\begin{aligned}d_1(\hat{R}_i^n(-m, s), \hat{R}_j^n(-m, s)) &\leq \sqrt{p} \left\| \frac{1}{n-1} \sum_{l \neq i} \rho_l^n(-m, s, \cdot) - \frac{1}{n-1} \sum_{l \neq j} \rho_l^n(-m, s, \cdot) \right\|_{L^1(\mathbf{T}^p)} = \\ \frac{\sqrt{p}}{n-1} \|\rho_j^n(-m, s, \cdot) - \rho_i^n(-m, s, \cdot)\|_{L^1(\mathbf{T}^p)} &\leq \frac{2}{n-1} \quad \forall s \in [-m, 0], \quad \forall i, j \in (1, \dots, n).\end{aligned}\tag{2.14}$$

By (2.8),

$$L_{c, Y^n, i}^n = \frac{1}{2} |\dot{x}|^2 - \langle c, \dot{x} \rangle - V(t, x) - \frac{n-1}{n} \int_{\mathbf{T}^p} W(x-y) \hat{\rho}_i^n(t, y) dy.$$

By lemma 1.1, we get the second inequality below.

$$\begin{aligned}\left\| V(t, x) + \frac{n-1}{n} \int_{\mathbf{T}^p} W(x-y) \hat{\rho}_i^n(t, y) dy \right\|_{C((-m, 0), C^3(\mathbf{T}^p))} &\leq \\ \|V(t, x)\| + \left\| \int_{\mathbf{T}^p} W(x-y) \hat{\rho}_i^n(t, y) dy \right\|_{C((-m, 0), C^3(\mathbf{T}^p))} &\leq C_1.\end{aligned}$$

As a result, the value function  $u_i^n$  satisfies point 2) of corollary 2.2; thus,  $Y_i^n \in Vect_m$  and we can apply lemma 2.3, getting that  $R_i^n$  belongs to  $Den_m^{reg}$ . Since this set is convex, (2.13) implies that  $\hat{R}_i^n \in Den_m^{reg}$ ; by lemma 2.4, we have that  $Den_m^{reg}$  is a compact set; thus, fixing  $i = 1$ , there is  $n_k \rightarrow +\infty$  such that  $\hat{R}_1^{n_k}$  converges to  $R \in Den_m^{reg}$ ; in particular,  $R$  and its density  $\rho$  satisfy (2.5). This gives convergence only for  $\hat{R}_1^{n_k}$ ; however, from (2.14) we get that

$$\sup_{i \in (1, \dots, n_k)} \sup_{s \in [-m, 0]} d_1(\hat{R}_i^{n_k}(-m, s), \hat{R}_1^{n_k}(-m, s)) \rightarrow 0 \quad \text{as } k \rightarrow +\infty\tag{2.15}$$

which implies that all  $\hat{R}_i^{n_k}$  converge to the same limit  $R$ . By the same argument of (2.14),

$$d_1(R^{n_k}(-m, s), \hat{R}_i^{n_k}(-m, s)) \leq \frac{1}{n}.$$

Thus, (2.15) implies the first formula of (2.11).

**Step 2.** We prove the convergence of the solutions of Hamilton-Jacobi. We set

$$\begin{cases} W_i^{n_k}(s, x) := \frac{n-1}{n} \int_{\mathbf{T}^p} W(x-y) \hat{\rho}_i^{n_k}(-m, s, y) dy = \frac{n-1}{n} \int_{\mathbf{T}^p} W(x-y) d\hat{R}_i^{n_k}(-m, s)(y) \\ \tilde{W}(s, x) := \int_{\mathbf{T}^p} W(x-y) \rho(-m, s, y) dy = \int_{\mathbf{T}^p} W(x-y) dR(-m, s)(y). \end{cases}\tag{2.16}$$

By (2.15) and the fact that  $d_1$  induces weak\* convergence, we get that

$$\sup_{i \in (1, \dots, n_k)} \|W_i^{n_k} - \tilde{W}\|_{C([-m, 0], C^3(\mathbf{T}^p))} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.\tag{2.17}$$

Now,  $u_i^{n_k}$  is the value function of  $L_{c,Y^{n_k},i}$ , whose potential is  $V(t,x) + W_i^{n_k}(t,x)$ ; by the last formula, we can apply point 3) of lemma 2.1 and get that  $u_i^{n_k}$  satisfies the limit in the second formula of (2.11), with  $u$  a solution of  $(HJ)^{V+\bar{W}}$  or, which is the same, of  $(HJ)_{\rho,f}$ .

**Step 3.** We prove that the limit density  $\rho$  solves  $(FP)_{-m,c-\partial_x u,\mu}$ .

From now on, for ease of notation, we drop the  $n_k$  of the subsequence. We recall that each  $\rho_i^n$  solves  $(FP)_{-m,Y_i^n,\delta_{z_i}}$ ; by the third formula of (2.11) and lemma 2.5, we get that, if  $\bar{\rho}_i$  is a solution of  $(FP)_{-m,c-\partial_x u,\delta_{z_i}}$ , then

$$\sup_i \|\rho_i^n \mathcal{L}^p - \bar{\rho}_i \mathcal{L}^p\|_{C^0([-m,0],\mathcal{M}_1(\mathbf{T}^p))} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Now (2.2) implies the inequality below, and the last formula implies the limit.

$$\begin{aligned} \|\rho^n \mathcal{L}^p - \frac{1}{n} \sum_{i=1}^n \bar{\rho}_i \mathcal{L}^p\|_{C([-m,0],\mathcal{M}_1(\mathbf{T}^p))} &= \left\| \frac{1}{n} \sum_{i=1}^n \rho_i^n \mathcal{L}^p - \frac{1}{n} \sum_{i=1}^n \bar{\rho}_i \mathcal{L}^p \right\|_{C([-m,0],\mathcal{M}_1(\mathbf{T}^p))} \leq \\ &\frac{1}{n} \sum_{i=1}^n \|\rho_i^n \mathcal{L}^p - \bar{\rho}_i \mathcal{L}^p\|_{C([-m,0],\mathcal{M}_1(\mathbf{T}^p))} \rightarrow 0. \end{aligned}$$

This means that  $\rho^n \mathcal{L}^p$  and  $\frac{1}{n} \sum_{i=1}^n \bar{\rho}_i \mathcal{L}^p$  have the same limit; we saw in step 1 that  $\rho^n \mathcal{L}^p$  converges to  $\rho \mathcal{L}^p$ ; thus, to prove that  $\rho \mathcal{L}^p$  solves  $(FP)_{-m,c-\partial_x u,\mu}$ , it suffices to prove that the limit of  $\frac{1}{n} \sum_{i=1}^n \bar{\rho}_i \mathcal{L}^p$  solves the same equation. This follows easily, since by definition  $\frac{1}{n} \sum_{i=1}^n \bar{\rho}_i \mathcal{L}^p$  solves the Fokker-Planck equation with drift  $c - \partial_x u$  and initial condition  $\frac{1}{n}(\delta_{z_1} + \dots + \delta_{z_n})$ , and (2.10) holds.

**Step 4.** We prove the last assertion of the lemma; the equality below comes from (2.6) and the fact that  $Y^n$  is minimal.

$$U^n(-m, (z_1, \dots, z_n)) = E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, X^n(-m, s, x), Y^n(s, X^n(-m, s, x))) ds \right\} + U(\rho^n(-m, 0, \cdot)).$$

We recall that, by corollary 2.7,

$$Y^n = (c - \partial_x u_1^n, \dots, c - \partial_x u_n^n).$$

Now  $\rho_i^n \mathcal{L}^p$  is the push-forward of the Wiener measure by  $X_i^n$ , and the Brownian motions  $w_i$  are independent. This implies that

$$\begin{aligned} U^n(-m, (z_1, \dots, z_n)) &= \frac{1}{n} \sum_{i=1}^n \int_{[-m,0] \times \mathbf{T}^p} L_c(t, x, c - \partial_x u_i^n) \rho_i^n(t, x) dt dx - \\ &\frac{1}{2n^2} \sum_{i \neq j \in \{1, \dots, n\}} \int_{[-m,0] \times \mathbf{T}^p \times \mathbf{T}^p} W(x_i - x_j) \rho_i^n(-m, t, x_i) \rho_j^n(-m, t, x_j) dx_i dx_j dt + \int_{\mathbf{T}^p} f(x) \rho^n(-m, 0, x) dx. \end{aligned}$$

Using (2.11), we get immediately that

$$U^n(-m, (z_1, \dots, z_n)) \rightarrow \int_{[-m,0] \times \mathbf{T}^p} \mathcal{L}_{c, \frac{1}{2}\rho}(t, x, c - \partial_x u) \rho(-m, 0, x) dt dx + \int_{\mathbf{T}^p} f(x) \rho(-m, 0, x) dx.$$

\\

**Proof of theorem 3.** Let the measure  $\mu$ , the couple  $(u, \rho)$  and the function  $U(-m, \mu)$  be as in the last lemma; let the operator  $\Lambda_c^m$  be as in the introduction, and let  $Y = c - \partial_x u$ . We are going to prove that

$$\begin{aligned} (\Lambda_c^m U)(\mu) &= U(-m, \mu) = E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\rho}(s, X, Y) ds + f(X(-m, 0, \mu)) \right\} = \\ &= \min_{\tilde{Y}} E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}(-m, 0, \mu)) \right\}. \end{aligned} \quad (2.18)$$

The functions  $\rho$  and  $\tilde{\rho}$  in the formula above satisfy  $(FP)_{-m, Y, \mu}$  and  $(FP)_{-m, \tilde{Y}, \mu}$  respectively, while  $X$  and  $\tilde{X}$  satisfy  $(SDE)_{-m, Y, \mu}$  and  $(SDE)_{-m, \tilde{Y}, \mu}$  respectively. The minimum is taken over all Lipschitz vector fields  $\tilde{Y}$ .

Note that, in principle,  $U(-m, \mu)$  could depend on the subsequence  $\{n_k\}_{k \geq 1}$  chosen in lemma 2.8; the formula above says that this is not the case. Moreover, it says that any  $(u, \rho)$  arising in lemma 2.8 as the limit of a subsequence, minimizes the last expression of (2.18).

The second equality of (2.18) follows from lemma 2.8: it is just another way of writing (2.12). Again by lemma 2.8,  $(u, \rho) \in \mathbf{S}$ , and thus, by the definition of  $(\Lambda_c^m U)(\mu)$ ,

$$(\Lambda_c^m U)(\mu) \leq E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\rho}(s, X, Y) ds + f(X(-m, 0, \mu)) \right\} = U(-m, \mu). \quad (2.19)$$

Now we prove that

$$U(-m, \mu) \leq \inf_{\tilde{Y}} E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}(-m, 0, z)) \right\}. \quad (2.20)$$

To prove this, we consider the  $n$ -particle value function  $U^n(-m, z_1, \dots, z_n)$ . Let  $\tilde{Y}$  be a Lipschitz vector field on  $[-m, 0] \times \mathbf{T}^p$ . Let  $\tilde{X}_i^n$  solve  $(SDE)_{-m, \tilde{Y}, \delta_{z_i}}$ ; let us suppose that (2.10) holds. Let us set  $\tilde{X}^n = (\tilde{X}_1^n, \dots, \tilde{X}_n^n)$  and  $\tilde{Y}^n = (\tilde{Y}, \dots, \tilde{Y})$ . Let  $\tilde{\rho}$  solve  $(FP)_{-m, \tilde{Y}, \mu}$  and let  $\tilde{\rho}_i$  solve  $(FP)_{-m, \tilde{Y}, \delta_{z_i}}$ ; by linearity, we get that  $\frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x)$  solves  $(FP)_{-m, \tilde{Y}, \frac{1}{n}(\delta_{z_1} + \dots + \delta_{z_n})}$ . In other words,  $\tilde{\rho}$  and  $\frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x)$  solve a Fokker-Planck equation with the same drift, but initial distributions  $\mu$  and  $\frac{1}{n}(\delta_{z_1} + \dots + \delta_{z_n})$  respectively; by (2.10), it is standard to see that

$$\sup_{s \in [-m, 0]} d_1 \left( \frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x) \mathcal{L}^p, \tilde{\rho}(-m, s, x) \mathcal{L}^p \right) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.21)$$

We set  $z = (z_1, \dots, z_n)$  and by (2.6) we get the inequality below; the first equality is the definition of  $L_c^n$ , the second one comes from the fact that the Brownian motions  $w_1, \dots, w_n$  are independent.

$$\begin{aligned} U^n(-m, z) &\leq \\ &= E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 L_c^n(s, \tilde{X}^n(-m, s, z), \tilde{Y}^n(s, \tilde{X}^n(-m, s, z))) ds + \frac{1}{n} \sum_{i=1}^n f(\tilde{X}_i(-m, 0, z_i)) \right\} = \\ &= E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \sum_{i=1}^n \int_{-m}^0 L_c(s, \tilde{X}_i, \tilde{Y}) ds - \frac{1}{2n^2} \sum_{i \neq j} \int_{-m}^0 W(\tilde{X}_i - \tilde{X}_j) ds + \frac{1}{n} \sum_{i=1}^n f(\tilde{X}_i(-m, 0, z_i)) \right\} = \end{aligned}$$

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \int_{[-m,0] \times \mathbf{T}^p} L_c(s, x, \tilde{Y}(s, x)) \tilde{\rho}_i(-m, s, x) ds dx - \\ & \frac{1}{2n^2} \sum_{i \neq j=1}^n \int_{[-m,0] \times \mathbf{T}^p \times \mathbf{T}^p} W(x_i - x_j) \tilde{\rho}_i(-m, s, x_i) \tilde{\rho}_j(-m, s, x_j) ds dx_i dx_j + \frac{1}{n} \sum_{i=1}^n \int_{\mathbf{T}^p} f(x) \tilde{\rho}_i(-m, 0, x) dx. \end{aligned}$$

We take limits in the formula above, using the last assertion of lemma 2.8 for the left hand side and (2.21) for the right hand side; we get that

$$U(-m, \mu) \leq \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}}(s, x, \tilde{Y}) \tilde{\rho}(t, x) dt dx + \int_{\mathbf{T}^p} f(x) \tilde{\rho}(-m, 0, x) dx.$$

Since  $\tilde{Y}$  is an arbitrary Lipschitz vector field, we get that (2.20) holds.

Let now  $(\bar{u}, \bar{\rho}) \in \mathbf{S}$  be minimal in the definition of  $(\Lambda_c^m U)(\mu)$ ; setting  $\tilde{Y} = c - \partial_x \bar{u}$ , (2.20) implies the inequality below, while the equality comes from our choice of  $\tilde{Y}$ .

$$U(-m, \mu) \leq E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}(-m, 0, z)) \right\} = (\Lambda_c^m U)(\mu).$$

This yields the inequality opposite to (2.19). In other words, we have proven the first equality of (2.18); the second one, as we have seen, is lemma 2.8. As for the third one, it suffices to prove the inequality opposite to (2.20), which we do presently.

Let  $(z_1, \dots, z_n)$  satisfy (2.10), let  $Y^n = (Y_1^n, \dots, Y_n^n)$  be minimal in (2.6), and let us set  $\tilde{Y}^n = Y_1^n$ . Let  $\tilde{\rho}^n$  satisfy  $(FP)_{-m, \tilde{Y}^n, \mu}$ . By (2.11) and (2.17), we obtain that there is  $\gamma_n \rightarrow 0$  such that

$$\begin{aligned} & E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}^n}(s, \tilde{X}^n, \tilde{Y}^n) ds + f(\tilde{X}^n(-m, 0, \mu)) \right\} \leq \\ & E_{w_1, \dots, w_n} \left\{ \int_{-m}^0 \left[ \frac{1}{n} \sum_{i=1}^n L_c(t, X_i^n, Y_i^n) + \frac{1}{2n^2} \sum_{i,j=1}^n W(X_i^n - X_j^n) \right] ds + \frac{1}{n} \sum_{i=1}^n f(X_i^n(-m, 0, \mu)) \right\} + \gamma_n. \end{aligned}$$

Since the limit of the term on the right is  $U(-m, \mu)$  by lemma 2.8, we get that

$$\inf_{\tilde{Y}} \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}\tilde{\rho}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}^n(-m, 0, \mu)) \right\} \leq U(-m, \mu)$$

yielding the inequality opposite to (2.20).

\\

We need the following lemma to prove the semigroup property.

**Lemma 2.9.** *Let  $Y_1$  be a Lipschitz vector field on  $[-(n+m), -n] \times \mathbf{T}^p$ , and let  $Y_2$  be a Lipschitz vector field on  $[-n, 0] \times \mathbf{T}^p$ . Then, for all  $\epsilon, \delta \in (0, 1)$ , there is a Lipschitz vector field  $Y$  which coincides with  $Y_1$  when  $t \in [-(n+m), -n]$ , and with  $Y_2$  when  $t \in [-n+\delta, 0]$ . Moreover,  $Y$  satisfies*

$$E_w \left\{ \int_{-(n+m)}^0 \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} \leq$$

$$\begin{aligned}
& E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_{Y_1}}(s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))) ds \right\} + \\
& E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_{Y_2}}(s, X_2(-n, s, \rho_{Y_1}(-n)), Y_2(s, X_2(-n, s, \rho_{Y_1}(-n)))) ds \right\} + \epsilon.
\end{aligned} \tag{2.22}$$

In the formula above,  $X_1$  solves  $(SDE)_{-(n+m), Y_1, \mu}$ ,  $X_2$  solves  $(SDE)_{-n, Y_2, \rho_1(-n)}$ ,  $X$  solves  $(SDE)_{-(n+m), Y, \mu}$  and  $\rho_1, \rho_2, \rho_Y$  are the densities of the laws of  $X_1, X_2$  and  $X$  respectively.

Moreover, we can require that

$$|U(\rho_Y(0)\mathcal{L}^p) - U(\rho_{Y_2}(0)\mathcal{L}^p)| \leq \epsilon. \tag{2.23}$$

**Proof.** Let  $\bar{\delta} \in (0, \delta)$ ; it is always possible to find a Lipschitz vector field  $Y$  coinciding with  $Y_1$  on  $[-(n+m), -n] \times \mathbf{T}^p$  and with  $Y_2$  on  $[-n + \bar{\delta}, 0] \times \mathbf{T}^p$ , and such that  $\|Y\|_\infty$  is bounded uniformly in  $\bar{\delta}$ ; we forego the easy proof of this fact.

We note that  $X = X_1$  when  $s \in [-(n+m), -n]$ , since both functions solve the same stochastic differential equation; as a consequence,

$$\begin{aligned}
& E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} = \\
& E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_{Y_1}}(s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))) ds \right\}.
\end{aligned}$$

Thus, it suffices to prove that

$$\begin{aligned}
& E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-n, s, \rho_{Y_1}(-n)), Y(s, X(-n, s, \rho_{Y_1}(-n)))) ds \right\} \leq \\
& E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_{Y_2}}(s, X_2(-n, s, \rho_{Y_1}(-n)), Y_2(s, X_2(-n, s, \rho_{Y_1}(-n)))) ds \right\} + \epsilon.
\end{aligned} \tag{2.24}$$

To prove this, we recall that  $X$  and  $X_2$  solve two stochastic differential equations with drift  $Y$  and  $Y_2$  respectively; this means that, for  $s \geq -n$  and any trajectory  $w$  of the Brownian motion, we have that

$$X(-n, s, \rho_{Y_1}(-n))(w) = X(-n, -n, \rho_{Y_1}(-n))(w) + \int_{-n}^s Y(\tau, X(-n, \tau, \rho_{Y_1}(-n)))(w) d\tau + w(s) - w(-n) \tag{2.25}$$

and

$$X_2(-n, s, \rho_{Y_1}(-n))(w) = X_2(-n, -n, \rho_{Y_1}(-n))(w) + \int_{-n}^s Y_2(\tau, X_2(-n, \tau, \rho_{Y_1}(-n)))(w) d\tau + w(s) - w(-n). \tag{2.26}$$

Since  $X_2(-n, -n, \rho_{Y_1}(-n))$  and  $X(-n, -n, \rho_{Y_1}(-n))$  have the same law  $\rho_{Y_1}(-n)$ , we can as well suppose that

$$X_2(-n, -n, \rho_{Y_1}(-n))(w) = X(-n, -n, \rho_{Y_1}(-n))(w) \tag{2.27}$$



for all realizations  $w$  of the Brownian motion. Subtracting (2.26) from (2.25) and using the formula above, we get the inequality below; the equality is the definition of the function  $a$ .

$$\begin{aligned} & |X(-n, s, \rho_{Y_1}(-n))(w) - X_2(-n, s, \rho_{Y_1}(-n))(w)| \leq \\ & \int_{-n}^s |Y(\tau, X(-n, \tau, \rho_{Y_1}(-n))(w)) - Y_2(\tau, X_2(-n, \tau, \rho_{Y_1}(-n))(w))| d\tau = \\ & \int_{-n}^s a[\tau, X(-n, \tau, \rho_{Y_1}(-n))(w), X_2(-n, \tau, \rho_{Y_1}(-n))(w)] d\tau \end{aligned}$$

Since  $Y$  and  $Y_2$  are bounded uniformly in  $\bar{\delta}$ , we get that  $|a| \leq M$  if  $\tau \in [-n, 0]$  for a constant  $M$  independent on  $\bar{\delta}$ ; since  $Y$  coincides with the Lipschitz  $Y_2$  on  $[-n + \bar{\delta}, 0]$ , we get that, for  $\tau \geq -n + \bar{\delta}$ ,

$$|a(\tau, x, y)| \leq K|x - y|$$

for a constant  $K$  independent on  $\bar{\delta}$ . From the last two formulas, we get that

$$\begin{aligned} & |X(-n, s, \rho_{Y_1}(-n))(w) - X_2(-n, s, \rho_{Y_1}(-n))(w)| \leq \\ & \int_{-n}^{-n+\bar{\delta}} M d\tau + \int_{-n+\bar{\delta}}^s K |X(-n, \tau, \rho_{Y_1}(-n))(w) - X_2(-n, \tau, \rho_{Y_1}(-n))(w)| d\tau. \end{aligned}$$

Using the Gronwall lemma and (2.27), we get that there is a function  $\gamma(\bar{\delta})$ , tending to zero as  $\bar{\delta}$  tends to zero, such that

$$|X(-n, s, \rho_{Y_1}(-n))(w) - \tilde{X}(-n, s, \rho_{Y_1}(-n))(w)| \leq \gamma(\bar{\delta})$$

for all realizations  $w$  of the Brownian motion. From this, (2.24) follows easily.

On the other hand, it is easy to see that the formula above implies that, as  $\bar{\delta} \rightarrow 0$ ,  $\rho_Y(0)\mathcal{L}^p$  converges weak\* to  $\rho_{Y_2}(0)\mathcal{L}^p$ . Since  $U$  is Lipschitz for the 1-Wasserstein distance, (2.23) follows.

\\

**Proposition 2.10.** 1) The map  $\Psi_c^m$  defined in the introduction has the semigroup property, i. e. for  $n, m \geq 0$  and  $U \in C(\mathcal{M}_1(\mathbf{T}^p), \mathbf{R})$ ,

$$\Psi_c^{n+m}U = \Psi_c^n \circ \Psi_c^m U.$$

2) If  $U \leq V \in C(\mathcal{M}_1(\mathbf{T}^p), \mathbf{R})$ , then  $\Psi_c^m U \leq \Psi_c^m V$ .

3) For all  $a \in \mathbf{R}$  and  $U \in C(\mathcal{M}_1(\mathbf{T}^p), \mathbf{R})$ ,  $\Psi_c^m(U + a) = (\Psi_c^m U) + a$ .

**Proof.** Properties 2) and 3) follow in a standard way from the definition of  $\Psi_c^m$ ; we prove 1).

Let  $\mu \in \mathcal{M}_1$ ; by the definition of  $\Psi_c^{n+m}U$  as an infimum, for any  $\epsilon > 0$  we can find a Lipschitz vector field  $Y$  such that

$$\Psi_c^{n+m}U(\mu) \geq E_w \left\{ \int_{-(n+m)}^0 \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} + U(\rho_Y(0)\mathcal{L}^p) - \epsilon$$

where  $X$  solves  $(SDE)_{-(n+m), Y, \mu}$  and  $\rho_Y$  is, as usual, the solution of the Fokker-Planck equation with initial condition  $\mu$ . By the Chapman-Kolmogorov formula, the formula above implies the first inequality below.

$$\begin{aligned}
(\Psi_c^{n+m}U)(\mu) &\geq E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} + \\
&E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-n, s, \rho_Y(-n)), Y(s, X(-n, s, \rho_Y(-n)))) ds \right\} + U(\rho_Y(0)\mathcal{L}^p) - \epsilon \geq \\
&E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} + (\Psi_c^n U)(\rho_Y(-n)\mathcal{L}^p) - \epsilon \geq \\
&(\Psi_c^m \circ \Psi_c^n U)(\mu) - \epsilon.
\end{aligned}$$

The second and third inequalities above come from the definition of  $\Psi_c^n U$  and  $\Psi_c^m \circ \Psi_c^n U$  as infima. Since  $\epsilon$  is arbitrary, this means that

$$(\Psi_c^{n+m}U)(\mu) \geq (\Psi_c^n \circ \Psi_c^n U)(\mu). \quad (2.28)$$

We prove the opposite inequality. By the definition of  $\Psi_c^m \circ \Psi_c^n(U)$ , we can find a Lipschitz vector field  $Y_1$  such that

$$\begin{aligned}
(\Psi_c^m \circ \Psi_c^n U)(\mu) &\geq E_w \left\{ \int_{-(m+n)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_{Y_1}}(s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))) ds \right\} + \\
&\Psi_c^n(U)(\rho_{Y_1}(-n)\mathcal{L}^p) - \epsilon.
\end{aligned} \quad (2.29)$$

By the definition of  $\Psi_c^n U$ , we can find another Lipschitz vector field  $Y_2$  such that

$$\begin{aligned}
\Psi_c^n U(\rho_{Y_1}(-n)) &\geq E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_{Y_2}}(s, X_2(-(n+m), s, \rho_{Y_1}(-n)), Y_2(s, X_2(-(n+m), s, \rho_{Y_1}(-n)))) ds \right\} + \\
&U(\rho_{Y_2}(0)\mathcal{L}^p) - \epsilon.
\end{aligned} \quad (2.30)$$

Let  $\epsilon, \delta > 0$ ; by lemma 2.9, we can find a Lipschitz vector field  $Y$  equal to  $Y_1$  on  $[-(n+m), -n] \times \mathbf{T}^p$  and to  $Y_2$  on  $[-n+\delta, 0] \times \mathbf{T}^p$ , such that (2.22) and (2.23) holds. The first inequality below comes from the definition of  $\Psi^{n+m}U$  as a infimum; the second one from (2.22) and (2.23); the third and fourth ones come from (2.30) and (2.29) respectively.

$$\begin{aligned}
(\Psi_c^{n+m}U)(\mu) &\leq E_w \left\{ \int_{-(n+m)}^0 \mathcal{L}_{c, \frac{1}{2}\rho_Y}(s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu))) ds \right\} + U(\rho_Y(0)\mathcal{L}^p) \leq \\
&E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_{Y_1}}(s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))) ds \right\} + \\
&E_w \left\{ \int_{-n}^0 \mathcal{L}_{c, \frac{1}{2}\rho_{Y_2}}(s, X_2(-n, s, \rho_{Y_1}(-n)), Y_2(s, X_2(-n, s, \rho_{Y_1}(-n)))) ds \right\} + U(\rho_{Y_2}(0)\mathcal{L}^p) + 2\epsilon \leq \\
&E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c, \frac{1}{2}\rho_{Y_1}}(s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))) ds \right\} + (\Psi_c^n U)(\rho_{Y_1}(-n)\mathcal{L}^p) + 3\epsilon \leq
\end{aligned}$$

$$(\Psi_c^n \circ \Psi_c^m U)(\mu) + 4\epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we get the inequality opposite to (2.28), and thus the thesis.

\\

### §3

#### Fixed points

As in [8] and in [15], the following proposition is essential in proving theorem 4.

**Proposition 3.1.** *Let  $U$  be linear as in theorem 2. Then, there is  $L > 0$ , independent on  $n$ , such that  $\Lambda_c^n U = \Psi_c^n U$  is  $L$ -Lipschitz for the Wasserstein distance  $d_1$ .*

To prove this proposition, we shall need two lemmas.

**Lemma 3.2.** *Let  $R \in C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$  and let  $u$  solve  $(HJ)_{R,f}$ . Then, there is  $C > 0$ , independent both of  $m \in \mathbf{N}$  and of  $R \in C([-m, 0], \mathcal{M}_1(\mathbf{T}^p))$ , such that*

$$\|\partial_x u(t, \cdot)\|_{C^1([-m, 0], C(\mathbf{T}^p))} + \|\partial_x u(t, \cdot)\|_{C([-m, 0], C^2(\mathbf{T}^p))} \leq C.$$

**Proof.** We have seen in section 1 that, if  $u$  is a solution of  $(HJ)_{R,f}$ ,  $v = e^{-\beta u}$  and  $a \in \mathbf{R}$ , then  $e^{-\beta a} v = e^{-\beta(u+a)}$  is a solution of  $(TS)_{R, e^{-\beta(f+a)}}$  with  $A = 0$ . Let  $a_k \in \mathbf{R}$  such that  $e^{-\beta a_k} v(-k, \cdot) = e^{-\beta(u+a_k)}$  satisfies (1.7) for  $k = 0, 1, 2, \dots$ . By the Feynman-Kac formula, for  $k \geq 0$ ,

$$e^{-\beta a_k} v(-k-1, \cdot) = L_{(\psi, 0, -1)}(e^{-\beta a_k} v(-k, \cdot)). \quad (3.1)$$

Since  $e^{-\beta a_k} v(-k, \cdot)$  satisfies (1.7), formulas (1.10) and (1.11) hold and we get that, for  $k \geq 0$ ,

$$\frac{1}{C_1} \leq e^{-\beta a_k} v(-k-1, x) \leq C_1 \quad \forall x \in \mathbf{T}^p.$$

We consider (3.1) with  $e^{-\beta a_k} v(-k-1, x)$  on the right hand side and differentiate under the integral sign; proceeding as in lemma 1.3, and using the last formula, we get that, for  $k \geq 0$ ,

$$\|e^{-\beta a_k} v(-k-2, \cdot)\|_{C^3(\mathbf{T}^p)} \leq C_2.$$

As in lemma 1.4, this implies that there is  $C_3 > 0$ , independent on  $k \geq 0$  (it depends only on  $C_1$  and  $C_2$ ) such that, for  $k \geq 0$ ,

$$\text{if } t \in [-(k+3), -(k+2)], \quad \text{then } \frac{1}{C_3} \leq e^{-\beta a_k} v(t, \cdot) \leq C_3 \quad \text{and}$$

$$\|e^{-\beta a_k} v(t, \cdot)\|_{C([-k+3], -(k+2)], C^3(\mathbf{T}^p))} + \|e^{-\beta a_k} v(t, \cdot)\|_{C^1([-k+3], -(k+2)], C^1(\mathbf{T}^p))} \leq C_3.$$

By our definition of  $v$ ,

$$\text{for } t \in [-(k+3), -(k+2)], \quad u = -\frac{1}{\beta} \log(e^{-\beta a_k v}) - a_k.$$

From the two formulas above, we get that

$$\|\partial_x u\|_{C([-m, -2], C^2(\mathbf{T}^p))} + \|\partial_x u\|_{C^1([-m, -2], C(\mathbf{T}^p))} \leq C \quad \text{for } t \leq -2.$$

It remains to bound  $\partial_x u(t, x)$  when  $t \in [-2, 0]$ ; since  $f \in C^3(\mathbf{T}^p)$ , this follows by differentiation under the integral sign in (1.20), and we are done.

\\

We recall some notation: in the following  $U^n(-m, z)$  will be the minimum in (2.6); moreover, given  $z = (z_1, \dots, z_n) \in (\mathbf{R}^p)^n$ , we set

$$|z|_1 = |z_1| + \dots + |z_n|$$

and we define  $z' \in (\mathbf{R}^p)^{n-1}$  by  $z = (z_1, \dots, z_n) = (z_1, z')$ .

**Lemma 3.3.** *Let  $U$  be as in theorem 2 and let  $U^n(-m, z)$  be defined as in (2.6). Then, there is a constant  $C > 0$  such that, for all positive integers  $n$  and  $m$ , we have*

$$|U^n(-m, z) - U^n(-m, \tilde{z})| \leq \frac{C}{n} |z - \tilde{z}|_1.$$

**Proof.** It suffices to prove that, for  $i = 1, \dots, n$ , the function

$$: z_i \rightarrow U^n(-m, z_1, \dots, z_i, \dots, z_n)$$

is  $\frac{C}{n}$ -Lipschitz and that the constant  $C$  does not depend neither on  $m \in \mathbf{N}$  nor on  $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in (\mathbf{T}^p)^{n-1}$ . We shall prove this for  $i = 1$ ; from this the general case follows, since  $U$  is a symmetric function of  $(z_1, \dots, z_n)$ .

We write the function  $U^n(-m, (z_1, z'))$  as in lemma 2.6, isolating particle  $z_1$ :

$$\begin{aligned} U^n(-m, (z_1, z')) &= E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^0 L_c(s, X_j(-m, s, z_j), Y_j(s, X_j(-m, s, z_j))) ds + \right. \\ &\quad \left. \frac{1}{2n^2} \sum_{j, i \neq 1} \int_{-m}^0 W(X_i(-m, s, z_i) - X_j(-m, s, z_j)) ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m, 0, z_j)) \right\} + \\ &\quad \frac{1}{n} E_{w_1} \left\{ \int_{-m}^0 L_{c, Y, 1}^n(s, X_1(-m, s, z_1), Y_1(s, X_1(-m, s, z_1))) ds + f(X_1(-m, 0, z_1)) \right\} \end{aligned} \quad (3.2)$$

where the vector field  $Y = (Y_1, \dots, Y_n)$  is minimal. By the definition of  $U^n(-m, (\tilde{z}_1, z'))$  as a minimum, we get that

$$U^n(-m, (\tilde{z}_1, z')) \leq E_{w_1, \dots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^0 L_c(s, X_j(-m, s, z_j), Y_j(s, X_j(-m, s, z_j))) ds + \right.$$

$$\begin{aligned} & \frac{1}{2n^2} \sum_{j,i \neq 1} \int_{-m}^0 W(X_i(-m, s, z_i) - X_j(-m, s, z_j)) ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m, 0, z_j)) \Big\} + \\ & \frac{1}{n} E_{w_1} \left\{ \int_{-m}^0 L_{c,Y,1}^n(s, X_1(-m, s, \tilde{z}_1), Y_1(s, X_1(-m, s, \tilde{z}_1))) ds + f(X_1(-m, 0, \tilde{z}_1)) \right\}. \end{aligned} \quad (3.3)$$

The term with  $E_{w_1, \dots, w_n}$  is identical in (3.2) and (3.3); defining  $u_1^n$  as in corollary 2.7, we see that the term with  $E_{w_1}$  is equal to the function  $u_1^n(-m, z_1)$  in (3.2), and to  $u_1^n(-m, \tilde{z}_1)$  in (3.3); subtracting (3.2) from (3.3), we get that

$$U^n(-m, (\tilde{z}_1, z')) - U^n(-m, (z_1, z')) \leq \frac{1}{n} [u_1^n(-m, \tilde{z}_1) - u_1^n(-m, z_1)] \leq \frac{C}{n} |\tilde{z}_1 - z_1|$$

where the last inequality comes from lemma 3.2. Exchanging the rôles of  $z_1$  and  $\tilde{z}_1$ , we get that the function  $: z_1 \rightarrow U^n(-m, (z_1, z'))$  is  $\frac{C}{n}$ -Lipschitz; we saw at the beginning of the proof that this implies the thesis.

\\

**Proof of proposition 3.1.** By lemma 2.8, we know that

$$\text{if } \frac{1}{n}(\delta_{z_1} + \dots + \delta_{z_n}) \rightarrow \mu \quad \text{then} \quad U^n(-m, z_1, \dots, z_n) \rightarrow U(-m, \mu).$$

We saw in (2.18) that  $U(-m, \mu) = (\Lambda_c^m U)(\mu)$ . Thus it suffices to show that

$$|U^n(-m, (x_1, \dots, x_n)) - U^n(-m, (y_1, \dots, y_n))| \leq L d_1 \left( \frac{1}{n}(\delta_{x_1} + \dots + \delta_{x_n}), \frac{1}{n}(\delta_{y_1} + \dots + \delta_{y_n}) \right).$$

It is standard ([5]) that

$$d_1 \left( \frac{1}{n}(\delta_{x_1} + \dots + \delta_{x_n}), \frac{1}{n}(\delta_{y_1} + \dots + \delta_{y_n}) \right) = \min_{\sigma} \frac{1}{n} \sum_{i=1}^n |x_i - y_{\sigma(i)}| \quad (3.7)$$

where the minimum is taken over all the permutations  $\sigma$  of  $\{1, \dots, n\}$ . In terms of transport, when we are connecting two  $n$ -uples of deltas, there is not just a minimal transfer plan, but a minimal transfer map.

Since

$$U^n(-m, (y_1, \dots, y_n)) = U^n(-m, (y_{\sigma(1)}, \dots, y_{\sigma(n)})),$$

we have to prove that, for  $\sigma$  minimal in (3.7),

$$|U^n(-m, (x_1, \dots, x_n)) - U^n(-m, (y_{\sigma(1)}, \dots, y_{\sigma(n)}))| \leq \frac{C}{n} \sum_{i=1}^n |x_i - y_{\sigma(i)}|.$$

But this is an immediate consequence of lemma 3.3.

\\

When  $U$  is linear, we define  $\Lambda_{c,\lambda} U = \Lambda_c U + \lambda$ ; thus, in case of a linear  $U$ , we have that  $\Lambda_{c,\lambda} U = \Psi_{c,\lambda} U$  for the operator  $\Psi_{c,\lambda}$  defined in the introduction. In the next lemma, we stick to the  $\Lambda_{c,\lambda}$  notation.

**Lemma 3.4.** *Let the operator  $\Lambda_{c,\lambda}$  be defined as in the introduction and let  $U = 0$ . Then, there is a unique  $\lambda \in \mathbf{R}$  such that*

$$\hat{U}(\mu) := \liminf_{n \rightarrow +\infty} (\Lambda_{c,\lambda}^n 0)(\mu)$$

*is finite for all  $\mu \in \mathcal{M}_1$ . Moreover,  $\hat{U}$  is  $L$ -Lipschitz for the constant  $L$  of proposition 3.1.*

**Proof.** Clearly, there is at most one  $\lambda \in \mathbf{R}$  for which the  $\liminf$  above is finite; let us prove that it exists. This means finding  $\lambda \in \mathbf{R}$  such that, for all  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ ,

$$-\infty < \liminf_{n \rightarrow +\infty} (\Lambda_{c,\lambda}^n 0)(\mu) < +\infty. \quad (3.8)$$

Note that the formula above implies that  $\hat{U}$  is finite; it is  $L$ -Lipschitz because it is the  $\liminf$  of  $L$ -Lipschitz functions.

By proposition 3.1,  $\Lambda_{c,0}^n 0$  is  $L$ -Lipschitz for all  $n \in \mathbf{N}$ ; since  $\mathcal{M}_1(\mathbf{T}^p)$  is a compact metric space, we can find  $M > 0$  such that

$$\max \Lambda_{c,0}^n 0 - \min \Lambda_{c,0}^n 0 \leq M \quad \forall n \geq 1. \quad (3.9)$$

Possibly taking a larger  $M$ , we can suppose that

$$\|\Lambda_{c,0}^1 0\|_{\sup} \leq M.$$

By point 2) of proposition 2.10, this implies the first inequality below; the equality follows by point 1), and the second inequality by point 3) of the same proposition.

$$(\Lambda_{c,0}^2 0)(\mu) = (\Lambda_{c,0}^1 (\Lambda_{c,0}^1 0))(\mu) \leq \Lambda_{c,0}^1 (0 + M) \leq 2M.$$

Exchanging signs, this implies that

$$\|\Lambda_{c,0}^2 0\|_{\sup} \leq 2M.$$

Iterating, we get

$$\|\Lambda_{c,0}^n 0\|_{\sup} \leq nM \quad \forall n \geq 1. \quad (3.10)$$

We set

$$a_n = \min_{\mu} (\Lambda_{c,0}^n 0)(\mu) \quad \text{and} \quad -\lambda = \liminf_{n \rightarrow +\infty} \frac{a_n}{n}. \quad (3.11)$$

From (3.10), it follows that  $\lambda \in [-M, M]$ . We assert that  $\lambda$  satisfies (3.8). We prove the inequality on the left of (3.8), since the one on the right is analogous; actually, we are going to prove that, for all  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and all  $n \in \mathbf{N}$ ,

$$(\Lambda_{c,\lambda}^n 0)(\mu) > -10M.$$

Indeed, let us suppose by contradiction that, for some  $m \in \mathbf{N}$  and  $\bar{\mu} \in \mathcal{M}_1(\mathbf{T}^p)$ , we have

$$(\Lambda_{c,\lambda}^m 0)(\bar{\mu}) \leq -10M.$$

By (3.9), this implies that, for all  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ ,

$$(\Lambda_{c,\lambda}^m 0)(\mu) \leq -9M. \quad (3.12)$$

Let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$ ; the first inequality below comes from (3.12) and points 1) and 2) of proposition 2.10, the equality from point 3) of the same proposition, the last inequality from (3.12).

$$(\Lambda_{c,\lambda}^{2m} 0)(\mu) \leq [\Lambda_{c,\lambda}^m(-9M)](\mu) = -9M + (\Lambda_{c,\lambda}^m 0)(\mu) \leq -18M.$$

Proceeding by induction, we find that

$$(\Lambda_{c,\lambda}^{km} 0)(\mu) \leq -9kM \quad \forall \mu \in \mathcal{M}_1(\mathbf{T}^p).$$

Since

$$(\Lambda_{c,\lambda}^{km} 0)(\mu) = (\Lambda_{c,0}^{km} 0)(\mu) + km\lambda,$$

we get that

$$(\Lambda_{c,0}^{km} 0)(\mu) \leq -9kM - (km)\lambda \quad \forall \mu \in \mathcal{M}_1(\mathbf{T}^p).$$

By the definition of  $\lambda$  in (3.11), this implies that  $-\lambda \leq -\lambda - \frac{9M}{m}$ ; this contradiction proves (3.8) and thus the lemma.

\\

**Proof of theorem 4.** Let  $\hat{U}$  be as in the last lemma; since  $\hat{U}$  may not be linear, we switch to the  $\Psi_{c,\lambda}^1$  notation. We have to prove that  $\hat{U}$  is a fixed point of  $\Psi_{c,\lambda}^1$  and that a minimizing vector field exists.

Let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and  $\epsilon > 0$ ; by the definition of  $\Psi_{c,\lambda}^1 \hat{U}$ , we can find a Lipschitz vector field  $\bar{Y}$  for which

$$E_w \left\{ \int_0^1 \mathcal{L}_{c,\frac{1}{2}\bar{\rho}}(s, \bar{X}, \bar{Y}) ds \right\} + \hat{U}(\bar{\rho}(1)\mathcal{L}^p) + \lambda \leq (\Psi_{c,\lambda}^1 \hat{U})(\mu) + \epsilon. \quad (3.13)$$

To use this formula, we are going to express  $\hat{U}(\bar{\rho}(1)\mathcal{L}^p)$  by the limit of lemma 3.4.

Let  $n \in \mathbf{N}$ ; by theorem 3, applied to  $f \equiv 0$  with an obvious translation in time, we can find  $Y_n$  be such that

$$(\Psi_{c,0}^n 0)(\mu) = \min_Y E_w \left\{ \int_1^{n+1} \mathcal{L}_{c,\frac{1}{2}\rho}(s, X, Y) \right\} = E_w \left\{ \int_1^{n+1} \mathcal{L}_{c,\frac{1}{2}\rho_n}(s, X_n, Y_n) \right\}$$

where  $\rho_n$  stays for  $\rho_{Y_n}$  and  $\rho$  for  $\rho_Y$ ; the initial time for (SDE) and (FP) is 1. Let  $\tilde{Y}$  be equal to  $\bar{Y}$  on  $[0, 1] \times \mathbf{T}^p$ , to  $Y_n$  on  $[1 + \delta, n + 1] \times \mathbf{T}^p$  and a Lipschitz connection in between. By lemma 2.9, we can choose the Lipschitz connection in such a way that

$$\begin{aligned} & E_w \left\{ \int_0^1 \mathcal{L}_{c,\frac{1}{2}\bar{\rho}}(s, \bar{X}(0, s, \mu), \bar{Y}(s, \bar{X}(0, s, \mu))) ds \right\} + \\ & E_w \left\{ \int_1^{n+1} \mathcal{L}_{c,\frac{1}{2}\rho_n}(s, X_n(1, s, \mu), Y_n(s, X_n(1, s, \rho_Y(1)))) ds \right\} \geq \end{aligned}$$

$$E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(s, \tilde{X}(0, s, \mu), \tilde{Y}(s, \tilde{X}(0, s, \mu))) ds \right\} - \epsilon. \quad (3.14)$$

Now (3.13) and the definition of  $\hat{U}$  imply the first inequality below, (3.14) the second one while the third one follows from the definition of  $\Psi_{c, \lambda}^{n+1}0$ . The equality at the end follows by the definition of  $\hat{U}$ .

$$\begin{aligned} (\Psi_{c, \lambda}^1 \hat{U})(\mu) &\geq E_w \left\{ \int_0^1 \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(s, \bar{X}(0, s, \mu), \bar{Y}) ds + \lambda \right\} + \\ \liminf_{n \rightarrow +\infty} E_w \left\{ \int_1^{n+1} \mathcal{L}_{c, \frac{1}{2}\rho_n}(s, X_n(1, s, \rho_{\bar{Y}}(1)), Y_n) ds + \lambda n \right\} - \epsilon &\geq \\ \liminf_{n \rightarrow +\infty} E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(s, \tilde{X}(0, s, \mu), \tilde{Y}) ds + \lambda(n+1) \right\} - 2\epsilon &\geq \liminf_{n \rightarrow +\infty} (\Psi_{c, \lambda}^{n+1}0)(\mu) - 2\epsilon = \hat{U}(\mu) - 2\epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we get that

$$(\Psi_{c, \lambda}^1 \hat{U})(\mu) \geq \hat{U}(\mu) \quad \forall \mu \in \mathcal{M}_1(\mathbf{T}^p). \quad (3.15)$$

On the other hand, let  $\mu \in \mathcal{M}_1(\mathbf{T}^p)$  and let  $Y_n$  minimize in the definition of  $(\Psi_{c, \lambda}^{n+1}0)(\mu)$ . Then, by the definition of  $\hat{U}$ , we get the equality below.

$$\hat{U}(\mu) = \liminf_{n \rightarrow +\infty} E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \frac{1}{2}\rho_n}(s, X_n(0, s, \mu), Y_n) ds + \lambda(n+1) \right\}$$

where  $\rho_n$  stays for  $\rho_{Y_n}$ . Let  $\{n_h\}$  be a subsequence on which the  $\liminf$  is attained. By lemma 3.2,  $Y_{n_h}$  is uniformly Lipschitz. Thus, we can apply Ascoli-Arzelà and, after further refining  $\{n_h\}$ , we can suppose that  $Y_{n_h}|_{[0,1] \times \mathbf{T}^p}$  converges to a vector field  $\bar{Y}$  in the  $C^0$  topology. The formula above yields the first equality below, while the second one follows from the fact that  $Y_{n_h}|_{[0,1] \times \mathbf{T}^p}$  converges uniformly to  $\bar{Y}$ ; the first inequality follows from the definition of  $\hat{U}(\bar{\rho}(1)\mathcal{L}^p)$ , the second one from the definition of  $\Psi_{c,0}^1$ .

$$\begin{aligned} \hat{U}(\mu) &= \liminf_{h \rightarrow +\infty} E_w \left\{ \int_0^1 \mathcal{L}_{c, \frac{1}{2}\rho_{n_h}}(s, X_{n_h}(0, s, \mu), Y_{n_h}) ds + \right. \\ &\quad \left. \int_1^{n_h+1} \mathcal{L}_{c, \frac{1}{2}\rho_{n_h}}(s, X_{n_h}(0, s, \rho_{n_h}(1)), Y_{n_h}) ds + \lambda(n_h+1) \right\} = \\ &\quad E_w \left( \int_0^1 \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(s, \bar{X}, \bar{Y}) ds + \lambda \right) + \\ &\quad \liminf_{h \rightarrow +\infty} E_w \left\{ \int_1^{n_h+1} \mathcal{L}_{c, \frac{1}{2}\rho_{n_h}}(s, X_{n_h}(1, 0, \rho_{n_h}(1)), Y_{n_h}) ds + \lambda n_h \right\} \geq \\ &\quad E_w \left( \int_0^1 \mathcal{L}_{c, \frac{1}{2}\bar{\rho}}(s, \bar{X}, \bar{Y}) ds + \lambda \right) + \hat{U}(\bar{\rho}(1)\mathcal{L}^p) \geq \Psi_{c,0}^1 \hat{U}(\mu) \quad \forall \mu \in \mathcal{M}_1(\mathbf{T}^p) \end{aligned}$$

where  $\bar{\rho}$  stays for  $\rho_{\bar{Y}}$ . This proves the inequality opposite to (3.15). Thus,  $\hat{U} = \Psi_{c,0}^1 \hat{U}$ ; by the last formula, this implies that  $\bar{Y}$  satisfies (5).

It remains to prove that the constant  $\lambda$  is unique. Let  $\Psi_{c, \lambda_1}^1 \hat{U}_1 = \hat{U}_1$  and  $\Psi_{c, \lambda_2}^2 \hat{U}_2 = \hat{U}_2$ . Let us suppose by contradiction that  $\lambda_1 < \lambda_2$ . Since  $\hat{U}_i$  is a continuous fixed point, we can suppose that  $\|\hat{U}_i\|_{\sup} \leq M$  for



$i = 1, 2$ ; as a consequence,  $\hat{U}_2 \geq \hat{U}_1 - 2M$ . By proposition 2.10, the first inequality below follows; the second equality follows from the fact that  $\hat{U}_1$  is a fixed point.

$$\begin{aligned}\Psi_{c,\lambda_2}^n \hat{U}_2 &\geq \Psi_{c,\lambda_2}^n \hat{U}_1 - 2M \geq \Psi_{c,\lambda_1}^n \hat{U}_1 + n(\lambda_2 - \lambda_1) - 2M = \\ &\hat{U}_1 + n(\lambda_2 - \lambda_1) - 2M \geq \hat{U}_2 + n(\lambda_2 - \lambda_1) - 4M.\end{aligned}$$

For  $n$  large enough, the last formula contradicts the fact that  $\hat{U}_2$  is a fixed point.

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